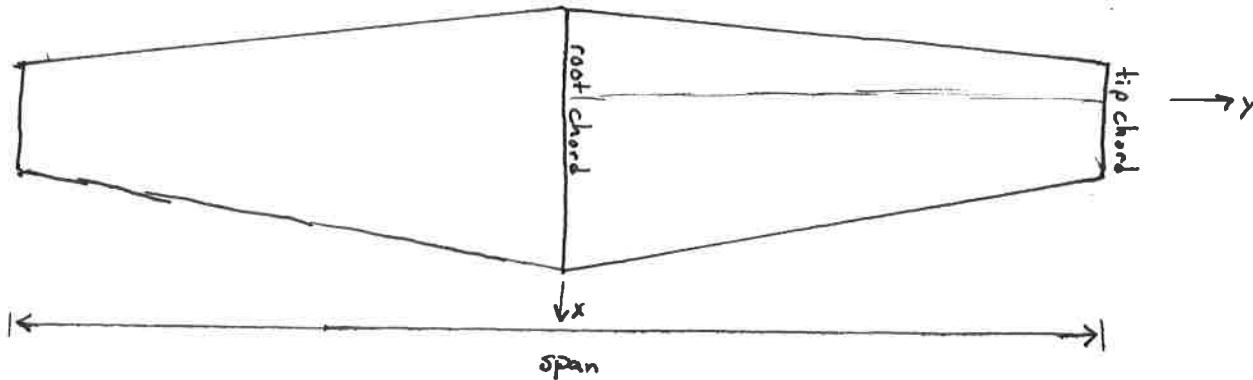


Lesson 21

Prandtl Lifting Line

Wings



A wing is described by:

$$\text{chord} \equiv c(y)$$

$$\Delta \equiv \text{Sweep} = \frac{1}{4}c$$

$$\text{Geometric twist} \equiv \alpha_g(y)$$

$$\text{Cross section} \equiv \text{Airfoil}(y)$$

$$\text{span} \equiv b$$

For a linearly tapered wing,

$$C_+ = \lambda C_r \quad \lambda \equiv \text{taper ratio}$$

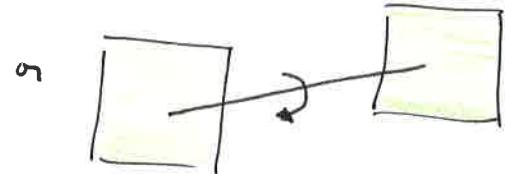
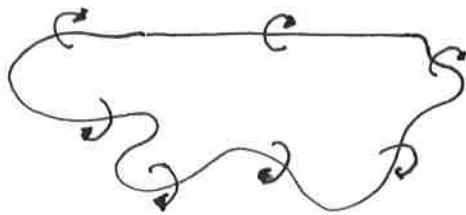
$$C(y) = C_r + \left(\frac{C_+ - C_r}{b/2} \right) y = C_r \left(1 + (\lambda - 1) \frac{2y}{b} \right)$$

$$\begin{aligned} \text{Area} \equiv S &= \int_{-b/2}^{b/2} C(y) dy = 2 \int_0^{b/2} C_r \left(1 + (\lambda - 1) \frac{2y}{b} \right) dy = 2C_r \left(y + \frac{\lambda-1}{6} 2y^2 \right) \Big|_0^{b/2} \\ &= C_r b \left(\frac{1}{2} + \frac{\lambda}{2} \right) = \bar{c} b \end{aligned}$$

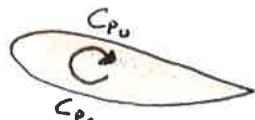
$$\text{Mean chord} \equiv \bar{c} = \frac{S}{b} = C_r \left(\frac{1}{2} + \frac{\lambda}{2} \right) \text{ at } y = \frac{1}{2} \left(\frac{b}{2} \right)$$

A vortex can not end in a fluid. T/F?

An inviscid vortex line must ^{start and} end on a surface or form a closed circuit.

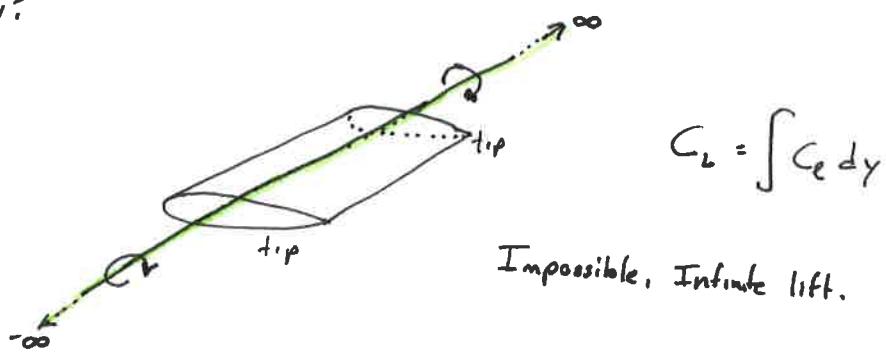


From 2D theory, an airfoil is known to create circulation (spanwise).

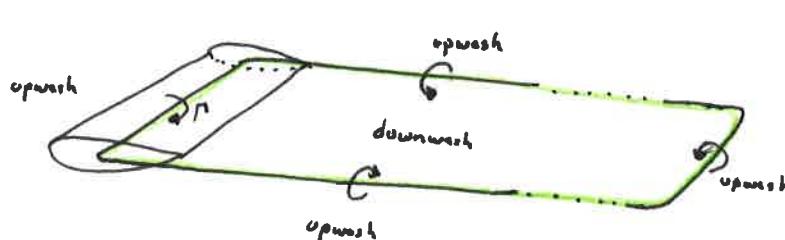


The lift C_L per unit span is $C_L = \frac{2\pi}{V}$. A differential $C_{p_u} - C_{p_e}$ exists.

Possibility?

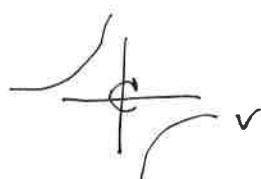


Possibility?

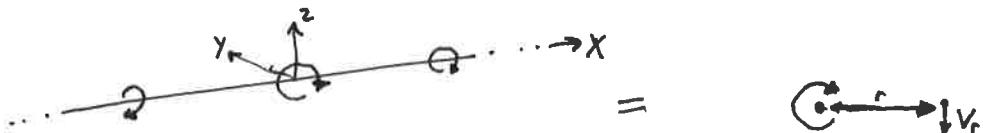


Lift is finite.

However, velocities at wingtips are infinite.

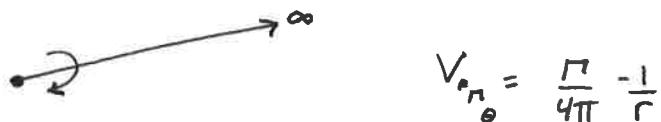


Wake consistent with a single vortex line from $-\infty$ to ∞

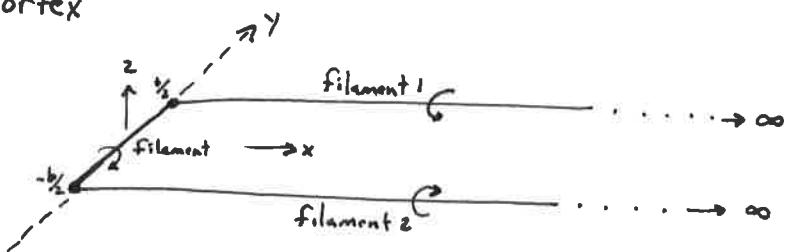


$$V_{r_0} = \frac{\Gamma}{2\pi} \cdot \frac{-1}{r}$$

So, a vortex filament from 0 to ∞ would give exactly half the full line's velocity.



Horseshoe vortex



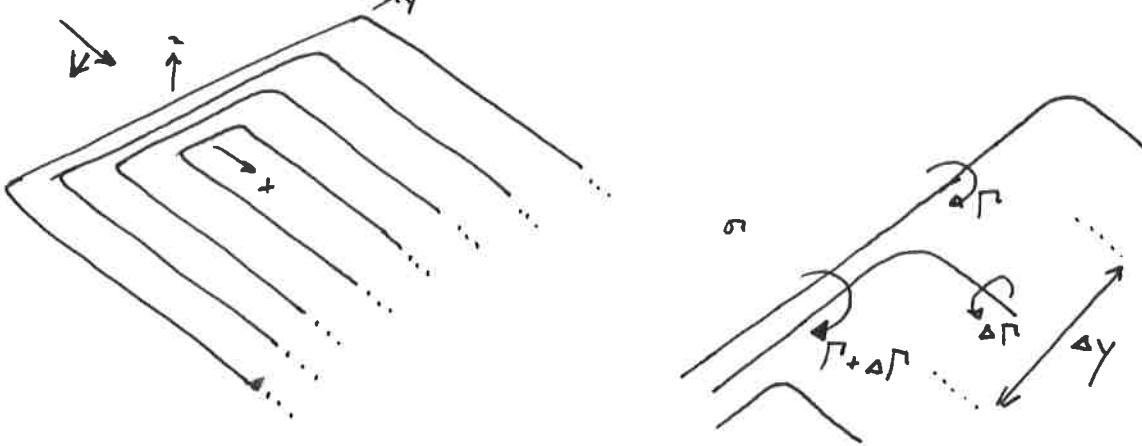
Downwash along y axis between $b_1/2$ and $b_2/2$ is $-V_{r_1} - V_{r_2}$:

$$w_y = \frac{\Gamma_1}{4\pi} \frac{1}{y - (-\frac{b_1}{2})} + \frac{\Gamma_2}{4\pi} \frac{1}{y - \frac{b_2}{2}}$$

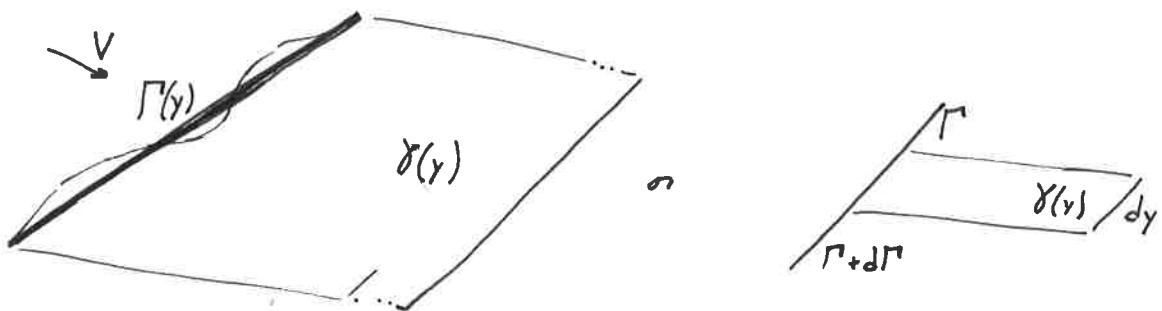
with $\Gamma_1 = \Gamma_2$

$$w_y = \frac{\Gamma}{4\pi} \left(\frac{1}{y + \frac{b_1}{2}} + \frac{1}{y - \frac{b_2}{2}} \right) = \frac{\Gamma}{4\pi} \frac{\frac{b_1 b_2}{4}}{y^2 - \frac{b_1^2 + b_2^2}{4}}$$

Stack discrete horseshoes along y-axis



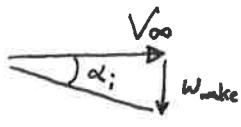
In the limit



$$\gamma(y) = -\frac{d\Gamma}{dy}$$

Downwash along y is

$$w_{wake}(y) = \frac{1}{4\pi} \int_{-b/2}^{b/2} \gamma(y') \frac{dy'}{y-y'} = \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy'} \frac{dy'}{y-y'} \quad \nwarrow \text{prin' value}$$



$$\alpha_i \approx -\frac{w_{wake}}{V_\infty}$$

For an unswept wing, lifting line theory uses 2D airfoil characteristics.

$$C_L = C_{L\alpha} (\alpha + \alpha_{aero} - \alpha_i)$$

↑ ↑ ↑
 rigid twist induced
 body and downward
 angle camber angle

$$\alpha_{aero} = \alpha_{geom} - \alpha_{zL}$$

From previous theory/derivation

$$L' = \rho V \Gamma \quad \text{and} \quad L' = \underbrace{\frac{1}{2} \rho V_m^2}_{q} C_L c$$

Such that

$$\rho V \Gamma = \frac{1}{2} \rho V^2 C_L c \quad \Rightarrow \quad \Gamma = \frac{1}{2} V C_L c$$

Substitute

$$\Gamma = \frac{1}{2} V c C_{L\alpha} (\alpha + \alpha_{geom} - \alpha_{zL} - \frac{1}{4\pi V} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy'} \frac{dy'}{y'-y})$$

Another integro-differential equation.

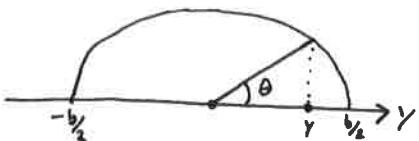
Remember that $\Gamma(y)$

$c(y)$

$\alpha_{geom}(y)$

$\alpha_{zL}(y)$

Fourier expansion and solution



$$y(\theta) = \frac{b}{2} \cos \theta$$

Expand Γ

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{\infty} A_n \sin(n\theta)$$

sin fits BCs: $\Gamma(\frac{b}{2}) = \Gamma(-\frac{b}{2}) = 0$

Expand deriv of Γ

$$\frac{d\Gamma}{dy'} dy' = \frac{d\Gamma}{d\theta} \frac{d\theta}{dy'} d\theta \frac{dy'}{d\theta} = \frac{d\Gamma}{d\theta} d\theta = 2bV_\infty \sum_{n=1}^{\infty} A_n \cos(n\theta) d\theta$$

Wake term

$$W_{\text{wake}} = \frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} 2bV_\infty \sum_{n=1}^{\infty} n A_n \cos(n\theta) \cdot \frac{1}{\frac{b}{2} \cos\theta' - \frac{b}{2} \cos\theta} d\theta'$$

$$= \frac{V_\infty}{\pi} \int_{\pi}^0 \underbrace{\sum_{n=1}^{\infty} \frac{n A_n \cos(n\theta')}{\cos\theta' - \cos\theta} d\theta'}_{\text{Remember this Glauert integral from TAT.}}$$

Switch order of \int and \sum

$$= \frac{V_\infty}{\pi} \sum_{n=1}^{\infty} n A_n \int_{\pi}^0 \underbrace{\frac{\cos n\theta'}{\cos\theta' - \cos\theta} d\theta'}_{\text{Remember this Glauert integral from TAT.}}$$

$$= -V_\infty \sum_{n=1}^{\infty} n A_n \frac{\sin(n\theta)}{\sin\theta}$$

Substitute into $\Gamma = \frac{1}{2} V_c C_{\ell\alpha} (\alpha + \alpha_{aero} - \alpha_i)$

$$2bV_\infty \sum A_n \sin(n\theta) = \frac{1}{2} V_c C_{\ell\alpha} (\alpha + \alpha_{aero} - \frac{V_m}{V_\infty} \sum n A_n \frac{\sin(n\theta)}{\sin\theta})$$

Rearrange to

$$\underbrace{\sum A_n \left(\sin n\theta + \frac{C(\theta)}{4b} C_{\ell\alpha} n \frac{\sin(n\theta)}{\sin\theta} \right)}_{\text{Interaction terms}} = \underbrace{\frac{C(\theta)}{4b} C_{\ell\alpha} (\alpha - \alpha_{aero}(\theta))}_{\text{Angles}}$$

Resembles form of panel method!

Solve for A_n given:

$$\begin{aligned} & C(\theta) \\ & \alpha \\ & \alpha_{aero}(\theta) \end{aligned}$$

Lift



$$L = \rho V \int_{-b/2}^{b/2} \Gamma dy = \rho V \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2b V_\infty \sum A_n \sin n\theta \left(-\frac{b}{2}\right) \sin \theta d\theta$$

$$= \frac{2b^2 V^2 \rho}{2} \int_0^\pi \underbrace{\sum A_n \sin n\theta \sin \theta d\theta}_{\text{From orthogonality}}$$

From orthogonality, $\int \sin n\theta \sin \theta d\theta \neq 0$ only when $n=1$

$$= \frac{1}{2} \rho V^2 b^2 \pi A_1, \quad \text{only 1 term!}$$

$$C_L = AR \pi A_1,$$

$$C_L = \frac{L}{gS} = \pi AR A_1,$$

Drag

$$D = -\rho \int_{-b/2}^{b/2} \Gamma w_{wake} dy = +\rho \int_{-b/2}^{b/2} 2b V_\infty \sum_{n=1}^{\infty} A_n \sin n\theta \cdot V_\infty \sum_n A_n \frac{\sin n\theta}{\sin \theta} dy$$

$$= \rho \int_{-\pi}^{\pi} 2b V_\infty \sum A_n \sin n\theta V_\infty \sum_n A_n \frac{\sin n\theta}{\sin \theta} -\frac{b}{2} \sin \theta d\theta$$

$$= \rho V^2 b^2 \sum_n A_n^2 \int_0^\pi \underbrace{\sin^2 \theta d\theta}_{\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax} \Rightarrow \frac{\pi}{2} - \frac{1}{4} \sin 2\pi x$$

$$\boxed{D = \frac{1}{2} \rho V^2 b^2 \pi \sum_n A_n^2}$$

Multiply by $1 = \left(\frac{L}{\frac{1}{2} \rho V^2 b^2 \pi A_1} \right)^2$

$$D = \frac{\left(\frac{L}{b}\right)^2}{\frac{1}{2} \rho V^2 \pi} \cdot \left(1 + 2 \left(\frac{A_2}{A_1}\right)^2 + 3 \left(\frac{A_3}{A_1}\right)^3 + 4 \cdot \left(\frac{A_4}{A_1}\right)^4 + \dots n \cdot \left(\frac{A_n}{A_1}\right)^n \right)$$

or in canonical form

$$D = \frac{\frac{L^2}{b^2}}{\frac{1}{2} \rho V^2 \pi} (1 + \delta) \quad \text{with} \quad \delta = \sum_{n=2}^{\infty} n \left(\frac{A_n}{A_1}\right)^2$$

or

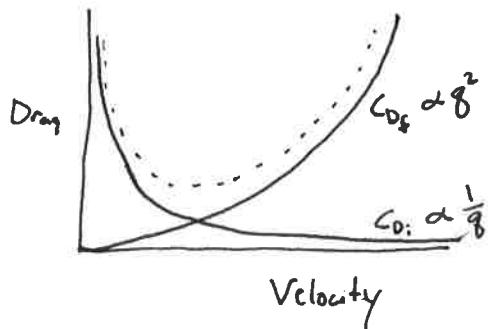
$$D = \frac{\left(\frac{L}{b}\right)^2}{\frac{1}{2} \rho V^2 \pi c} \quad \text{with} \quad c = \frac{1}{1 + \delta}$$

$$C_D = \frac{\left(\frac{L}{b}\right)^2}{g^2 \pi c}$$

$$C_D = \frac{D}{g S} = \frac{\pi}{AR} \sum n A_n^2 = \frac{C_L^2}{\pi AR c}$$

\uparrow
Notice $\frac{\text{Span loading}}{(\text{dynamic pressure})^2}$

This form is particularly useful



Drag:

$$D = \frac{\left(\frac{L}{b}\right)^2}{\frac{1}{2} \rho V^2 \pi} (1+\delta) \quad \text{but} \quad \frac{L^2}{b^2 S} = C_L^2$$

$$C_D = \frac{C_L^2 S^2 (1+\delta)}{\pi b^2 S} \quad \frac{C_L^2}{\pi AR} (1+\delta) = \frac{C_L^2}{\pi AR e}$$

$$\boxed{C_D = \frac{C_L^2}{\pi AR e}}$$

Induced drag is proportional to C_L^2 for a given AR

$$C_D = \frac{C_L^2}{\pi AR e} = \frac{C_L^2 S}{\pi b^2 e} \cdot \frac{S}{S} = \frac{C_L^2 \bar{c}^2}{\pi e S}$$

And on and on ...

Optimizc.

$$C_D = \frac{C_L^2}{\pi AR} (1 + \delta)$$

$$\delta = 2 \left(\frac{A_2}{A_1} \right)^2 + 3 \left(\frac{A_3}{A_1} \right)^2 + \dots$$

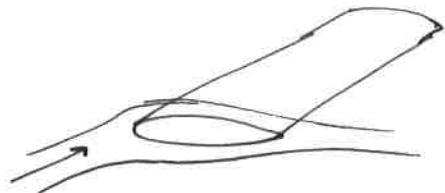
Set $A_2 = A_3 = A_4 = \dots A_N = 0$

$$A_1 \neq 0$$

Wake

$$W_{\text{wake}} = -V_\infty \sum n A_n \frac{\sin(n\theta)}{\sin\theta} = -V_\infty A_1 = -V_\infty \frac{C_L}{AR\pi}$$

$$\alpha_i = -\frac{W_{\text{wake}}}{V} = \frac{C_L}{\pi AR}$$



Gamma:

$$\Gamma(y) = \frac{1}{2} V c C_{L\alpha} (\alpha + \alpha_{\text{geom}} - \alpha_i)$$

which is from the derived eqn.

$$2b V_\infty A_1 \sin \theta = \frac{1}{2} V c(b) C_{L\alpha} (\alpha + \alpha_{\text{geom}} - \alpha_i)$$