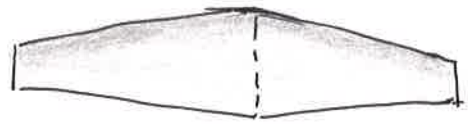
Lesson 16

Prandtl Lifting Line

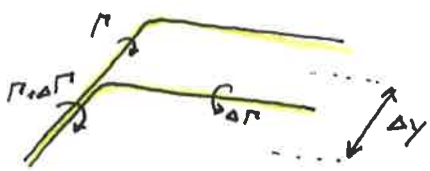
Prandtl Lifting Line

Assumptions:

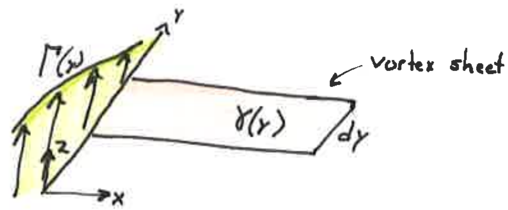
- Locally 2D chordwise flow
- No sweep
- No dihedral (flat)
- AR > 4
- α in linear C_{ℓ_α} region (we will discuss another approach using experimental data!)



Wake strength



$\Rightarrow$



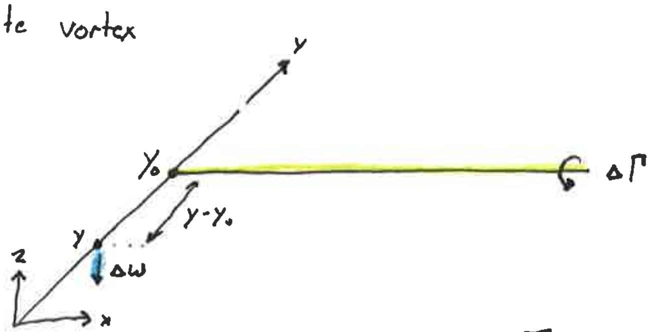
$$\frac{-(\Gamma + \Delta\Gamma) + (\Gamma)}{\Delta y} = \Delta\gamma \quad \Rightarrow \quad \boxed{\frac{d\Gamma}{dy} = -\gamma}$$

$$\frac{\Delta\Gamma}{\Delta y} = \Delta\gamma$$

$$\boxed{\gamma = -\frac{d\Gamma}{dy}}$$

A decrease in the bound vortex must be accompanied by an increase in the wake vortex (to satisfy Helmholtz)

Single semi infinite vortex



What is the velocity Δw in the (-) z direction consistent/induced from a semi-infinite vortex of strength $\Delta\Gamma$?

$$V = \frac{\Gamma}{4\pi h}$$

$V \leftarrow \Delta w$, $\Gamma \leftarrow \Delta\Gamma$, $h \leftarrow y - y_0$

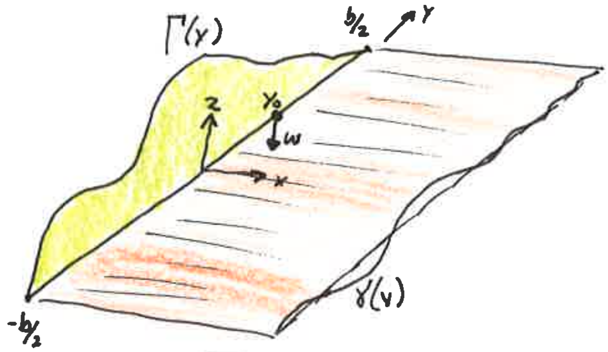
$$\Delta w = \frac{-\Delta\Gamma}{4\pi(y_0 - y)}$$

In the limit

$$\frac{dw}{dy} = \frac{-1}{4\pi} \cdot \frac{1}{y_0 - y} \cdot \frac{d\Gamma}{dy} \quad \Rightarrow \quad \boxed{dw = \frac{-1}{4\pi(y_0 - y)} \frac{d\Gamma}{dy} dy}$$

Notice that this is only valid on the y axis (x = 0)

Vertical Velocity at a single point resulting/consistent/imposed/induced by a distribution of $\Gamma(y)$ from a wing



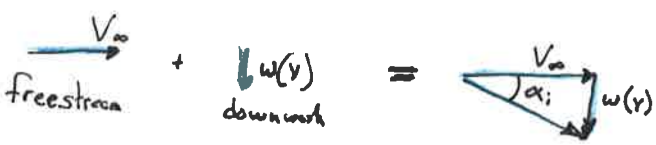
Integrate dw across the wing span

$$w(y_0) = \int dw$$

$$= \int_{-b/2}^{b/2} -\frac{1}{4\pi(y-y_0)} \frac{d\Gamma}{dy} dy$$

$$\boxed{w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y-y_0}\, dy}$$

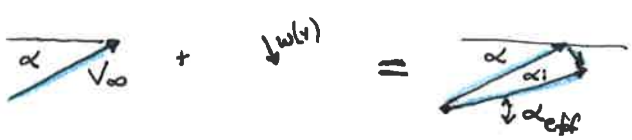
Local angle of attack.



$\alpha_i \equiv$ Induced Angle of Attack

$$= -\operatorname{atan}\left(\frac{w(y)}{V_\infty}\right) \approx \frac{-w(y)}{V_\infty}$$

What about a freestream at an angle of attack α?



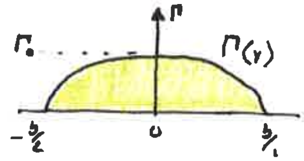
Effective angle of attack

$$\alpha_{eff} = \alpha - \alpha_i$$

$$\alpha_{eff}(y) = \alpha - \frac{w(y)}{V_\infty}$$

Varies with y

Elliptical Lift Distribution



$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y}{b/2}\right)^2} = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

Lift

$$L = \int_{-b/2}^{b/2} L'\, dy = \int_{-b/2}^{b/2} \rho V_\infty \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}\, dy$$

if $y = \frac{b}{2}\cos\theta$ and $dy = -\frac{b}{2}\sin\theta\, d\theta$

$$= \rho V_\infty \Gamma_0 \int_{\pi}^{0} \sqrt{1 - \cos^2\theta}\,\left(-\frac{b}{2}\sin\theta\right) d\theta = \rho V_\infty \Gamma_0 \left(-\frac{b}{2}\right) \int_{\pi}^{0} \sin^2\theta\, d\theta$$

$\sin^2 + \cos^2 = 1$

$$= \rho V_\infty \Gamma_0 \left(-\frac{b}{2}\right) \underbrace{\left.\frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta \right|_{\pi}^{0}}_{-\frac{\pi}{2}}$$

$$= \frac{1}{2}\rho V_\infty \Gamma_0 \frac{b}{2}\pi$$

Downwash

$$\frac{d\Gamma}{dy} = \frac{d}{dy}\left(\Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}\right) = \frac{-4\frac{y}{b}\Gamma_0}{\sqrt{b^2 - 4y^2}}$$

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y - y_0}\, dy = \frac{-4}{4\pi}\frac{\Gamma_0}{b}\int_{-b/2}^{b/2} \frac{y}{\sqrt{b^2 - 4y^2}}\,\frac{1}{y - y_0}\, dy$$

Substitute $y = \frac{b}{2}\cos\theta$

$$= \frac{-4}{4\pi}\frac{\Gamma_0}{b}\int_{\pi}^{0} \frac{\frac{b}{2}\cos\theta}{\underbrace{\sqrt{b^2 - 4\frac{b^2}{4}\cos^2\theta}}_{b\sqrt{1-\cos^2\theta} \to \sin\theta}} \cdot \frac{1}{\frac{b}{2}\cos\theta - \frac{b}{2}\cos\theta_0}\; \frac{-b}{2}\sin\theta\, d\theta$$

$$= \frac{\Gamma_0}{\pi b}\int_{\pi}^{0} \frac{1}{2}\,\frac{\cos\theta}{\cos\theta - \cos\theta_0}\, d\theta$$

Remember this integral?

Glauert Integral

$$\int_0^{\pi} \frac{\cos n\theta \, d\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

$$w(y_0) = \frac{-\Gamma_0}{\pi b 2}\int_0^{\pi} \frac{\cos\theta}{\cos\theta - \cos\theta_0}\, d\theta = \frac{-\Gamma_0}{2\pi b}\,\frac{\pi \overset{\sin\theta_0}{\cancel{\sin n\theta_0}}}{\sin\theta_0} = -\frac{\Gamma_0}{2b}$$

$\boxed{\text{The downwash is a constant for an elliptical lift distribution}}$

Induced angle due to Lift

$$\alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty} \quad \text{and} \quad L = \frac{1}{4}\rho V_\infty \Gamma_0 b \pi$$

solve for $\Gamma_0 = \dfrac{4L}{\rho V_\infty b \pi}$

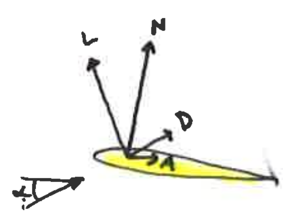
$$= \frac{4L}{\rho V_\infty b\pi} \cdot \frac{1}{2bV_\infty}\frac{\frac{1}{2}}{\frac{1}{2}} = \underbrace{\frac{L}{\frac{1}{2}\rho V_\infty^2 b^2 \pi}}$$

$$C_L \equiv \frac{L}{\frac{1}{2}\rho V_\infty^2 S} \Rightarrow C_L S = \frac{L}{\frac{1}{2}\rho V_\infty^2}$$

$$= \frac{C_L S}{\pi b^2} = \frac{C_L}{\pi AR}$$

$$\boxed{\alpha_i = \frac{C_L}{\pi AR}}$$

elliptical only



For small α, $L \approx N$ since $\cos\alpha \approx 1$

and $D \approx \sin\alpha \approx \alpha$ since $\alpha \ll 1$

Thus, $D \approx L\alpha_i$

$$\boxed{C_D \approx C_L \alpha_i = \frac{C_L^2}{\pi AR}}$$

Drag due to lift, "induced drag", depends on lift squared and inversely with aspect ratio.

Geometry of Elliptical Lift Distributions

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y \cdot 2}{b}\right)^2} \quad \text{and} \quad \Gamma_0 = \frac{4L}{\rho V_\infty b \pi}$$

$$= \frac{4L}{\rho V_\infty b \pi} \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

Also,

$$\underset{\substack{\text{lift}\\\text{per}\\\text{span}}}{L'} = \underset{\substack{\text{lift}\\\text{coeff}}}{C_\ell} \cdot \underset{\substack{\text{dynamic}\\\text{pressure}}}{q} \cdot \underset{\text{chord}}{c(y)} \quad \Rightarrow \quad c(y) = \frac{L'}{C_\ell q}$$

Since α_i is constant for an elliptical wing, C_ℓ is only $C_\ell(\alpha)$ (not a function of y)

$$C_\ell = 2\pi(\alpha - \alpha_i - \alpha_{ZL})$$

Also

$$L' = \rho V \Gamma(y) = \rho V \frac{4L}{\rho V b \pi} \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$$= \frac{4L}{b\pi} \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

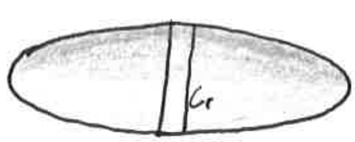
Thus

$$c(y) = \frac{L'}{C_\ell q} = \frac{4L}{b\pi} \sqrt{1 - \left(\frac{2y}{b}\right)^2} \, \frac{1}{2\pi(\alpha - \alpha_i - \alpha_{ZL})} \cdot \frac{1}{\frac{1}{2}\rho V_\infty^2}$$

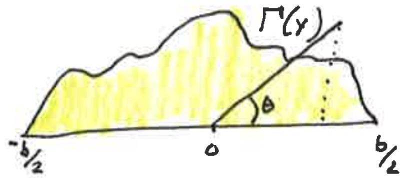
The only y variation is $\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$c(y) = \sqrt{1 - \left(\frac{2y}{b}\right)^2} \cdot c_r$$

This describes an ellipse.



General Lift Distribution



Expand $\Gamma(y)$ in terms of Fourier components $\qquad y = \frac{b}{2}\cos\theta \qquad dy = -\frac{b}{2}\sin\theta \, d\theta$

$$\Gamma(\theta) = 2bV_\infty \underbrace{\sum_{n=1}^{\infty} A_n \sin(n\theta)}_{A_1 \sin\theta \; + A_2 \sin 2\theta \; + A_3 \sin 3\theta \; + A_4 \sin 4\theta}$$

Notice that we picked $\sin n\theta$ which <u>always</u> fits the boundary conditions of $\Gamma(0) = \Gamma(\pi) = 0$

Downwash equ needs $d\Gamma/dy$

$$\frac{d\Gamma}{dy}dy = \left(\frac{d\Gamma}{d\theta}\frac{d\theta}{dy}\right)\left(d\theta \frac{dy}{d\theta}\right) = \frac{d\Gamma}{d\theta}d\theta$$

$$= 2bV_\infty \sum_{n=1}^{\infty}\left(A_n \cos(n\theta)\, n\right) d\theta$$

Wake velocity

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y - y_0}dy = \underbrace{-\frac{1}{4\pi}\int_{\pi}^{0} \frac{2bV_\infty \sum(A_n \cos n\theta \, n)}{\frac{b}{2}\cos\theta - \frac{b}{2}\cos\theta_0}d\theta}_{\text{Did you catch the trick? Amazing!}}$$

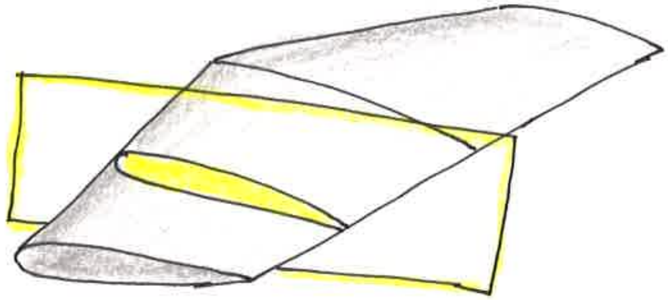
$$= \frac{V_\infty}{\pi}\int_0^{\pi} A_n n \frac{\cos n\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty}{\pi}\sum_{i=1}^{\infty} nA_n \underbrace{\int_0^{\pi}\frac{\cos n\theta}{\cos\theta - \cos\theta_0}d\theta}_{\text{Glauert Integral}}$$

$$\frac{\pi \sin n\theta_0}{\sin\theta_0}$$

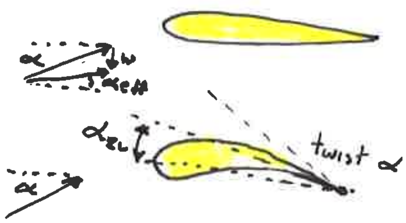
$$= \frac{V_\infty}{\pi}\pi \sum_{i=1}^{\infty} nA_n \frac{\sin n\theta_0}{\sin\theta_0}$$

$$\boxed{w(\theta_0) = V_\infty \sum_{i=1}^{\infty} nA_n \frac{\sin n\theta_0}{\sin\theta_0}}$$

$$\alpha_i = \frac{w}{V_\infty}$$



On this cross section,



$$L' = \rho V \Gamma$$

$$\underbrace{\frac{L'}{q c}}_{C_l} = \frac{\rho V \Gamma}{\frac{1}{2}\rho V^2 c}$$

$$C_l = \frac{2\Gamma}{Vc}$$

$$\Gamma = \frac{1}{2} V c \, C_l$$

Effective AOA $= \alpha + \underbrace{\alpha_{aero}}_{\alpha_{geo} + \alpha_{ZL}} - \alpha_i$

Expand Γ w/ Fourier series

$$\Gamma = \underbrace{2bV_\infty \sum_{n=1}^{\infty} A_n \sin(n\theta)}_{\text{Circulation at a particular point } \theta \text{ in the wing}} = \frac{1}{2} V_\infty \underbrace{c(\theta)}_{\text{chord function wrt } \theta} \underbrace{C_l}_{\text{lift coefficient}}(\alpha, \alpha_{aero}, \alpha_i)$$

Linear Aerodynamics

$$C_l = 2\pi(\alpha + \alpha_{aero} - \alpha_i)$$

Expand

$$2bV_\infty \sum_{n=1}^{\infty} A_n \sin(n\theta) = \frac{1}{2} V_\infty c(\theta) \underbrace{2\pi}_{C_{l_\alpha}} \left(\alpha + \alpha_{aero}(\theta) - \sum_{n=1}^{\infty} n A_n \frac{\sin n\theta}{\sin\theta} \right)$$

Rearrange

$$\underbrace{\sum_{n=1}^{\infty} A_n \left(\sin(n\theta) + \frac{c(\theta)}{4b} C_{l_\alpha} n \frac{\sin n\theta}{\sin\theta} \right)}_{\text{freestream + downwash}} = \underbrace{\frac{c(\theta)}{4b} C_{l_\alpha} (\alpha + \alpha_{aero})}_{\text{is tangent to the local effective airfoil}}$$

Solve for A_n given $c(\theta)$, $\alpha_{aero}(\theta)$, α, b

Lift

$$L = \int L'\, dy = \int \rho V \Gamma\, dy = \rho V_\infty \int_\pi^0 2bV_\infty \sum (A_n \sin n\theta)\left(-\tfrac{b}{2}\sin\theta\, d\theta\right)$$

$$= \frac{2b^2 V_\infty^2 \rho}{2} \int_\pi^0 -\underbrace{\left(\sum A_n \sin n\theta\right) \sin\theta}_{\text{orthogonal, so } \underline{\text{only}}\ n=1 \text{ contributes!!}}\, d\theta$$

$$= b^2 V_\infty^2 \rho \frac{\pi}{2} A_1$$

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S} = \frac{b^2 V_\infty^2 \rho\, \pi/2\, A_1}{\frac{1}{2}\rho V^2 S} = \pi AR\, A_1$$

Lift only depends on the A_1 term.

What $\underline{\text{is}}$ the A_1 term? $\quad \Gamma = 2bV_\infty A_1 \sin\theta$

Convert to y coordinate

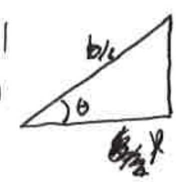
$$\Gamma = 2bV_\infty A_1 \sin\left(\operatorname{acos}\left(\tfrac{2y}{b}\right)\right)$$

$y = \frac{b}{2}\cos\theta$

$\frac{2y}{b} = \cos\theta$

$\theta = \operatorname{acos}\left(\frac{2y}{b}\right)$

$$\Gamma = 2bV_\infty A_1 \underbrace{\sin\left(\operatorname{acos}\left(\tfrac{2y}{b}\right)\right)}_{\sqrt{\left(\frac{b}{2}\right)^2 - y^2}}$$



$h = \sin\theta \cdot b/2$

$y = \cos\theta \cdot b/2$

$\operatorname{acos}\left(\frac{y2}{b}\right)$

$\sin\left(\operatorname{acos}\left(\frac{y2}{b}\right)\right) = \frac{h}{b/2}$

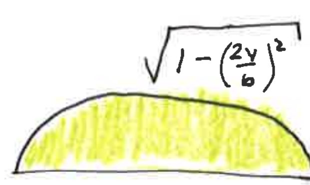
$h^2 + y^2 = \frac{b^2}{4}$

$h = \sqrt{\left(\frac{b}{2}\right)^2 - y^2}$

$$= 2bV_\infty A_1 \sqrt{\left(\tfrac{b}{2}\right)^2 - y^2} \cdot \frac{2}{b} \cdot \frac{b}{2}$$

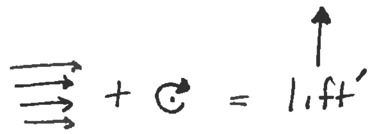
$$= 2bV_\infty A_1 \sqrt{1 - \left(\tfrac{4y^2}{b^2}\right)}\ \frac{b}{2}$$

$$= b^2 V_\infty A_1 \underbrace{\sqrt{1 - \left(\tfrac{2y}{b}\right)^2}}_{\underline{\text{Elliptical Distribution}}}$$

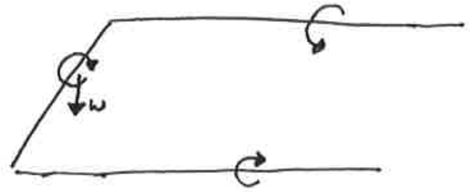


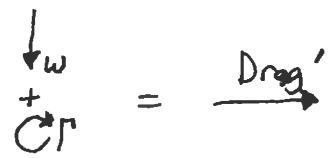
$$\boxed{\text{The overall lift is the elliptical portion of } \Gamma}$$

Drag



What if we consider drag' as the force in the perpendicular direction from the vertical perturbation/downwash velocity?





$$\text{Drag}' \approx w\Gamma$$

Integrate D' over span.

$$D = \int D' dy = \int_{-b/2}^{b/2} w\Gamma\, dy = \int_{\pi}^{0} V_\infty \left(\sum_{i=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta}\right)\left(2bV_\infty \sum_{i=1}^{\infty} A_n \sin n\theta\right) \frac{-b}{2}\sin\theta\, d\theta$$

$$= V_\infty^2 b^2 \underbrace{\int_0^{\pi} \left(\sum_{i=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta}\right)\left(\sum_{i=1}^{\infty} A_n \sin n\theta\right) \sin\theta\, d\theta}_{\text{term by term orthogonal}}$$

$$\boxed{D = \frac{1}{2}\rho V^2 b^2 \pi \sum_{i=1}^{\infty} nA_n^2}$$

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 S} = \frac{\frac{1}{2}\rho V^2 b^2 \pi \sum nA_n^2}{\frac{1}{2}\rho V^2 S} = AR\,\pi \sum nA_n^2$$

$$C_L = \pi\, AR\, A_1$$

$$C_D = AR\,\pi \left[\sum n\left(\frac{A_n}{A_1}\right)^2\right] \cdot A_1^2 = AR\,\pi \left[\sum n\left(\frac{A_n}{A_1}\right)^2\right] \frac{C_L^2}{\pi^2 AR^2}$$

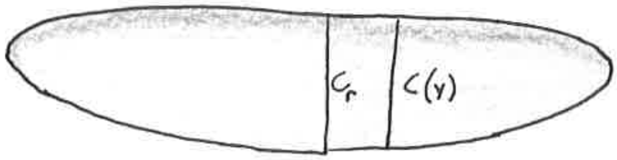
Drag depends on C_L^2 and inversely with AR and with square of non-elliptical terms

$$\boxed{C_D = \frac{1}{\pi AR} C_L^2 (1+\delta)}$$

$$\delta = \sum_{n=2}^{\infty} n\left(\frac{A_n}{A_1}\right)^2$$

$$1+\delta \equiv \frac{1}{e}$$

Flat Elliptical Wing



- Elliptical
- No twist "flat"

$$c(\theta) = c_r \sin\theta$$

$$c(y) = c_r \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$$S = \frac{\pi}{4} b c_r$$

$$AR = \frac{4}{\pi} \frac{b}{c_r}$$

$$A_1 = \frac{\frac{c_{l_\alpha}}{\pi AR}}{1 + c_{l_\alpha}/\pi AR} \cdot \left(\alpha + \alpha_{zero}\right)$$

$$\boxed{C_{L_\alpha} = \frac{c_{l_\alpha}}{1 + \frac{c_{l_\alpha}}{\pi AR}}}$$

$$\boxed{C_{D_i} = \frac{C_L^2}{\pi AR}}$$

Oswald Efficiency Factor

$$C_D = \frac{1}{\pi AR} \cdot \frac{1}{e} \cdot C_L^2$$

Oswald Eff' factor $\equiv e \equiv \dfrac{1}{1+\delta}$

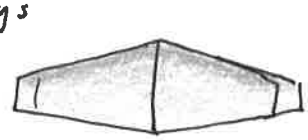
"e" measures the efficiency with respect to induced drag as compared to an elliptical wing

$e = 1$

This is a powerful approach valid for more than lifting line assumptions

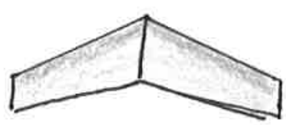
Straight wings

From: Aircraft Design, Raymer



$$e \approx 1.78\left(1 - 0.045\, A^{0.68}\right) - 0.64$$

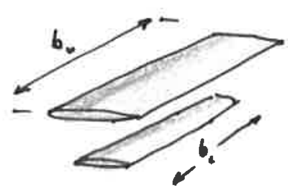
Swept wings



$$e \approx 4.61\left(1 - 0.045\, A^{0.68}\right)\left(\cos \Lambda_{LE}\right)^{0.15} - 3.1$$

$$\Lambda_{LE} > 30°$$

Biplanes



$$e \approx \frac{\mu^2 (1 + r^2)}{\mu^2 + 2\sigma\mu r + r^2}$$

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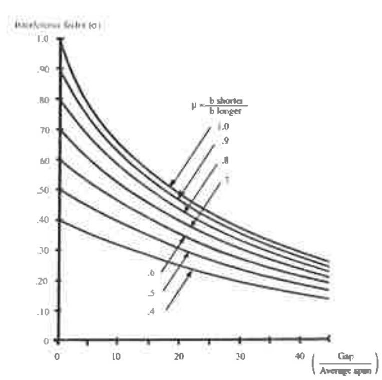


Fig. 12.34 Prandtl's biplane interference factor (Ref. 12).

$\mu = \dfrac{\text{shorter span}}{\text{longer span}}$

$\sigma =$ Interference factor depending on $\dfrac{\text{gap}}{\text{span}}$

$r = \dfrac{\text{lift on shorter}}{\text{lift on longer}}$

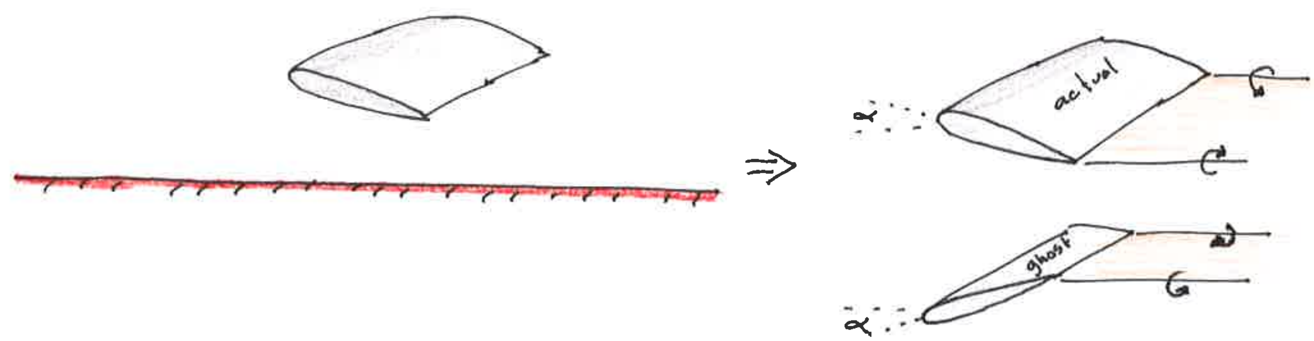
$$AR = \frac{\max(b_u, b_l)}{S_u + S_l}$$

Ground Effect:

Near the ground, a wing becomes <u>more</u> efficient.

$$C_D = K_{eff} C_L^2 \quad \text{with} \quad \frac{K_{eff}}{K} = \frac{33\left(\frac{h}{b}\right)^{1.5}}{1 + 33\left(\frac{h}{b}\right)^{1.5}}$$

We can model this with a ghost image of a wing below the surface

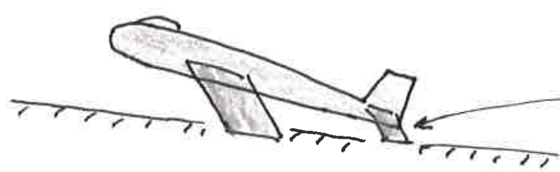


- In effect (pun intended), the downwash is prevented from passing through the ground
- Or, consider the ghost image has an upwash on the actual wing

We saw this in the tiny.cc/AEM313 Wing Vortex video, where the vortex moved outboard ~~[illegible]~~ as the aircraft approached the ground.

Ground effect can be a serious issue for the unsuspecting pilot (especially in low wing Piper a/c)

Ground effect can affect S+C at takeoff and landing



The tail is nearer the ground as the aircraft rotates for takeoff

Ground effect allows for high speed flying boats (Ground Effect Vehicle)

"Ekranoplan" tiny.cc/AEM313 Ground Effect