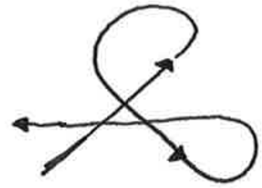
Cloverleaf

Airspeed calibration for high speed aircraft with no reference/visual to the ground and arbitrary airspeeds + headings.

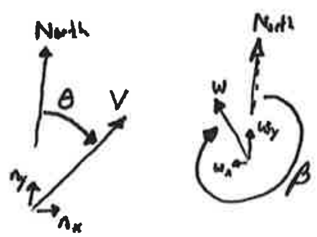


We have 3 unknowns:

- KIAS → KCAS "correction"
- Wind speed
- Wind direction

Ground speed:

$$\vec{G} = \vec{V}_{true} + \vec{W} = \underbrace{|V_{true}|\begin{pmatrix} n_x & n_y \end{pmatrix} + |W|\begin{pmatrix} w_x & w_y \end{pmatrix}}_{\text{Vector components in N and E directions}} = |G|\begin{pmatrix} g_x & g_y \end{pmatrix}$$



Absolute value

$$|G|^2 = \Big(\underbrace{|V_t| n_x + |W| w_x}_{x\text{-component}}\Big)^2 + \Big(\underbrace{|V_t| n_y + |W| w_y}_{y\text{-component}}\Big)^2$$

$$= (V n_x + W w_x)^2 + (V n_y + W w_y)^2$$

$n_x = \sin\theta$ $\quad w_x = \sin\beta\,(-1)$

$n_y = \cos\theta$ $\quad w_y = \cos\beta\,\underbrace{(-1)}_{\text{wind direction is "from" not towards.}}$

$$= V^2 n_x^2 + W^2 w_x^2 + 2VW\, n_x w_x + V^2 n_y^2 + W^2 w_y^2 + 2VW\, n_y w_y$$

$w_x^2 + w_y^2 = 1$

Airspeed:

$V_{true} = f(V_{cas})$

$= f(V_{ias} + \text{correction})$

$$G^2 = V^2 + W^2 + 2VW(n_x w_x + n_y w_y) = V^2 + W^2 + 2VW(\sin\theta\sin\beta + \cos\theta\cos\beta)$$

$$\boxed{G^2 = f(V_{ias} + c)^2 + W^2 + 2f(V_{ias} + c)W(\sin\theta\sin\beta + \cos\theta\cos\beta)}$$

this has to be true for any airspeed, direction, wind, etc.

GPS:

From a GPS, we can find groundspeed. We can read off Vias and heading θ and atmosphere terms to calc $f(\,)$

Solution:

$$\underset{\text{known}}{G^2} = \underset{\text{atmo}}{f}(\underset{\text{known}}{V_{ias}} + \underset{\text{unknown ①}}{c})^2 + \underset{\text{unknown ③}}{W^2} + 2f(V_{ias} + c)W(\underset{\text{known}}{\sin\theta}\,\underset{\text{unknown ③}}{\sin\beta} + \cos\theta\cos\beta)$$

3 unknowns so we need 3 equations (i.e. 3 independent lines)

A cloverleaf!

Numerical solution (one possibility):

$$\begin{pmatrix} U(x,y,z) \\ V(x,y,z) \\ W(x,y,z) \end{pmatrix} = \begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix} + \underbrace{\begin{bmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dV}{dx} & \frac{dV}{dy} & \frac{dV}{dz} \\ \frac{dW}{dx} & \frac{dW}{dy} & \frac{dW}{dz} \end{bmatrix}}_{\text{Jacobian} = J} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} + \text{higher order terms}$$

If we set the LHS to zero, we can solve for the $\Delta x, \Delta y, \Delta z$ that makes this true

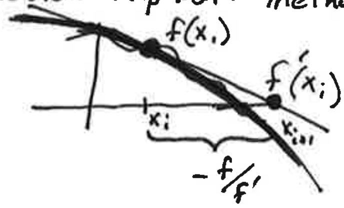
$$0 = \begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix} + [J] \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \Rightarrow \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = -[J]^{-1} \begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}$$

Then correct the x, y, z guess by $\Delta x \; \Delta y \; \Delta z$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i+1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i + \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i - [J]^{-1} \begin{pmatrix} U_0(x_i \; y_i \; z_i) \\ V_0 \quad \ldots \\ W_0 \quad \ldots \end{pmatrix}$$

This is the multidimensional equivalent of Newton-Raphson's method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



Really fast convergence to zero.... when it works...

Residual:

$$R \equiv f(V_{ias} + c)^2 + W^2 - 2 f(V_{ias} + c) \, W \left(\sin\theta \sin\beta + \cos\theta \cos\beta\right)$$

States are: W, β, c

$\frac{dR}{dW}$ is NOT too hard to find, but $\frac{dR}{dc}$ is really not analytically possible. $\frac{d R(f(V_{ias}+c)^2)}{dc} = ?$:-(

We can use numerical derivatives $\frac{dR}{dc} \approx \frac{R(c+\delta c) - R(c - \delta c)}{2\delta c}$ where δc is a small const ≈ 0.001 or so

so now you have

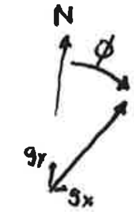
$$\begin{pmatrix} R_{line\,1} \\ R_{line\,2} \\ R_{line\,3} \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}}_{\text{above derivation}} \quad \text{and} \quad \begin{bmatrix} \frac{dR_1}{dw} & \frac{dR_1}{d\beta} & \frac{dR_1}{dc} \\ \frac{dR_2}{dw} & \ddots & \vdots \\ \frac{dR_3}{dw} & \cdots & \frac{dR_3}{dc} \end{bmatrix} = J$$

Iterate:

Iterate several times to find W, β, c

Cloverleaf Ground Track

"Use ground track rather than heading"



$$|G|(g_x \;\; g_y) = |V|(n_x \;\; n_y) + |W|(w_x \;\; w_y)$$

solve for V term

$$|V|(n_x \;\; n_y) = |G|(g_x \;\; g_y) - |W|(w_x \;\; w_y) = \left(G g_x - W w_x \,,\; G g_y - W w_y\right)$$

square

$$V^2 = G^2 g_x^2 + W^2 w_x^2 - 2GW g_x w_x + G^2 g_y^2 + W^2 w_y^2 - 2GW g_y w_y$$

$$= G^2 + W^2 - 2GW\left(g_x w_x + g_y w_y\right)$$

So the residual equation is

$$R = G^2 + W^2 - V^2 - 2GW\left(g_x w_x + g_y w_y\right)$$

$$= G^2 + W^2 - f(V_{ias} + c)^2 + 2GW\left(\sin\phi \sin\beta + \cos\phi \cos\beta\right)$$

Why might this be better?

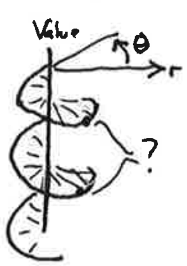
- GPS gives ground track to deg
- True heading is not so simple (magnetic declination, instrument bias, drift, visual resolution)

$\boxed{\text{Huge advantage}}$

However, this implementation does not work well! Why?

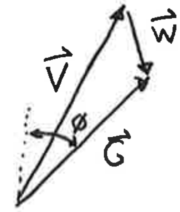
- The wind vector is the culprit. When $W = 0$, the jacobian is rank deficient (i.e. not invertible)
- $\frac{d}{dx}(xy) = y$, but if $y = 0$, then $\frac{d(xy)}{dx}$ is zero for any x. Optimize?
- The math reason is that we are saying that $|W|$ and β are independent, but this is not true

  If the wind is 270 at 2 kts and then $\Delta kt = -3$ kt, what is the wind vector?
  Is this an analytic function?



Avoid using this polar representation

Cloverleaf Generic $\bar{w}$ GPS



$$\underbrace{|G|(g_x \;\; g_y)}_{\text{groundspeed vector}} = \underbrace{|V|(n_x \;\; n_y)}_{\text{true airspeed vector}} + \underbrace{|W|(w_x \;\; w_y)}_{\text{wind vector}}$$

Using a GPS:

We know $|G|$, track and airspeed, the unknowns are wind velocity and direction.

However, for math reasons, we <u>don't</u> want to use wind direction, but rather wind components.

We also don't want to use the true airspeed <u>direction</u> (i.e. less precision than actual gnd track).

$$|V|(n_x \;\; n_y) = |G|(g_x \;\; g_y) - \underbrace{|W|(w_x \;\; w_y)}_{\text{write as } (W_x \;\; W_y)}$$

Find magnitude.

$$V^2 = (G g_x - W_x)^2 + (G g_y - W_y)^2$$

Find a residual function to drive to zero.

$$R = (G g_x - W_x)^2 + (G g_y - W_y)^2 - V^2$$

~~Add~~ Substitute for ground track direction,

$$g_x = \sin\phi \qquad g_y = \cos\phi$$

$$R = (G\sin\phi - W_x)^2 + (G\cos\phi - W_y)^2 - V^2$$

Substitute for IAS to TAS

$$\boxed{R = (G\sin\phi - W_x)^2 + (G\cos\phi - W_y)^2 - f(\overset{IAS+C}{IAS}, alt, temp)^2}$$

Numerical:

Newton Raphson iteration

$$\overset{\text{drive to zero}}{\vec{\varnothing}} = \begin{pmatrix} U(x,y,\dots) \\ V(x,y,\dots) \\ \vdots \end{pmatrix} = \begin{pmatrix} U_0 \\ V_0 \\ \vdots \end{pmatrix} + \begin{bmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{d\dots} \\ \frac{dV}{dx} & & \\ \vdots & \ddots & \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \vdots \end{pmatrix} + \text{higher order terms}$$

Solving for $\vec{U} = 0$

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \vdots \end{pmatrix} = -[J]^{-1} \underbrace{\begin{pmatrix} U_0 \\ V_0 \\ \vdots \end{pmatrix}}_{\text{residual vector}}$$

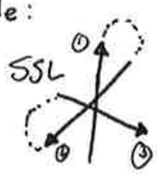
Update

$$\begin{pmatrix} W_x \\ W_y \\ C \end{pmatrix} = -[J] \begin{pmatrix} W_x \\ W_y \\ C \end{pmatrix}$$

and $\frac{dU}{dx} \approx \frac{U(x+\delta) - U(x-\delta)}{2\delta}$ with $\delta \ll$

Repeat 10 - 100 times or set a terminating criteria

Example:



① Heading $\approx$ N, Track 10°, 112 kt groundspeed, IAS is 115 kts

③ Heading $\approx$ SE, Track 116°, 128 kt groundspeed, 115 kt IAS

② Heading $\approx$ NW, Track 233, 88 kt groundspeed, 110 kt IAS

$\Rightarrow$ Wind 270 @ 20, C is -5 kts

Example 2:

Heading	KIAS	GS	Alt	OAT
358.2	105	162.8	14500	0
113.6	102	127.3	14600	5
248.4	100	132.5	14200	2

$\Rightarrow$ Recovers: Wind 165 at 20, C is -5 kts