GES 554

Partial Differential Equations

Lesson 1

Introduction

Skills Quiz: Friday
- closed book, closed notes, no calculator
- 20-25 minutes
- 10 one-line questions

Logistics:
- Lecture volume ok?
- Seating sufficient?
- Any other issues?

Terminology and Common Operators

- Derivatives $\frac{df}{dx} \equiv \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$$\frac{df}{dx} = f_x = \partial_x f = f'$$

Avoid f' unless clear what derivative is wrt.

- Gradient $\nabla f = \frac{df}{dx}\hat{\imath} + \frac{df}{dy}\hat{\jmath} + \frac{df}{dz}\hat{k}$

Scalar input, Vector output

- Divergence $\operatorname{div} F = \frac{dF}{dx} + \frac{dF}{dy} + \frac{dF}{dz}$

- Laplacian $\Delta f = \nabla^2 f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2}$

- Curl

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \equiv \hat{\imath}\left(\partial_y F_3 - \partial_z F_2\right) - \hat{\jmath}\left(\partial_x F_3 - \partial_z F_1\right) + \hat{k}\left(\partial_x F_2 - \partial_y F_1\right)$$

- Solution

A function that satisfies the governing equation

If at any point you are unsure of your solution, <u>test it!</u>

Classification

PDE order

The order of the highest partial derivative

- 1st order $\frac{df}{dx}$ term
- N^{th} order $\frac{d^nf}{dx^n}$ term

Examples:

Q: What order is the diffusion equation?

A: The diffusion equation is $u_t - D u_{xx} = 0$ $\boxed{2^{nd} \text{ order}}$

1st deriv'

2nd deriv'
Highest deriv

Q: What order is the transport equation $u_t - V u_x = 0$?

A: $\boxed{1^{st} \text{ order}}$ the highest derivative is a 1st deriv.

Q: What is a 0^{th} order PDE?

A: Algebra?!

Q: Why does this matter?

A: Stay tuned.

Linear / Non-Linear PDE

- A linear PDE has an algebraic linear combination of terms.

$$Au_t + Bu_x + Cu_x + Du_{xxx} + \sin(t)\, u_z = Z$$

- Non-linear is everything else
  - Multiplication of terms containing states.

    $u \cdot u_x \qquad u_x^2 \qquad \sqrt{u_x^3}$

  - Most functions of states

    $\sin(u) \qquad \sin(u_{xx})$

  - Anything that changes the functional "shape" of the u term.

- Either linear or non-linear. Although, sometimes we talk about the linear part and the non linear part of a single PDE

$$\underbrace{Au_t + Bu_x}_{\text{linear part}} + \underbrace{C\sin(u_x) + Du_{xx}^4}_{\text{non-linear part}} = Z$$

Q: Is the momentum conservation equation below linear or non-linear?

$$\frac{d}{dt}(\rho u) + \frac{d}{dx}F(\rho, u) = 0$$

A: Depends on $F(\rho, u)$

Q: Give / Find an $F(\rho, u)$ making the above equation i) linear and ii) non-linear.

A: linear $F(\rho, u) = c\rho u$ non-linear $F(\rho, u) =$ Anything else.

Why?!?!

Classification

- Number of variables. ≡ # independent variables (states)

  $U_t + U_{xx}$ has 2 variables (t, x)

  $U_t + U_{xx} + U_{yy}$ has 3 variables (t, x, y)

  $U_t + aU_{xx}$ has 2 variables, a is a constant.

- Homogeneous and non-homogeneous.

  Are the non-state terms zero? homogeneous.

- 2^{nd} Order, 2 variable, linear canonical PDE

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$$

  - Parabolic when $B^2 - 4AC = 0$

    Ex: Heat diffusion $(B = C = 0)$ and substitute $t = y$

$$U_t + AU_{xx} = 0$$

  - Hyperbolic when $B^2 - 4AC > 0$

    Ex: Wave equation

$$BU_{tt} + A_{xx} = 0$$

  - Elliptic when $B^2 - 4AC < 0$

    Ex: steady state stress in a plate (approx).

$$U_{xx} + U_{yy} = 0$$

Canonical Forms

- Canonical $\begin{cases}\text{Latin "rule" as in Canon law} \\ \text{Greek "reed"}\end{cases}$

- Standard convention of writing PDEs in a consistent way

- Group derivative terms to left hand side. Constants on RHS.

$$Au_t + Bu_x + Cu_{xx} = Z$$

- For numerical work, the canonical form is modified (sometimes) to place the time derivatives (temporal) on the LHS and the space derivatives (spatial) on the RHS. We will see why later.

$$u_t = -\frac{B}{A}u_x - \frac{C}{A}u_{xx} + \frac{Z}{A}$$

- A canonical form is usually non-dimensionalized

$$U_t = A\,U_{xx} \qquad \text{where} \quad \begin{matrix} U\ [K] \\ t\ [s] \\ x\ [m] \end{matrix} \quad \text{or} \quad \begin{matrix} U\ [\frac{slug}{m^3}] \\ t\ [s] \\ x\ [ft] \end{matrix}$$

$$U^* = \frac{U}{U_o} \qquad t^* = \frac{t}{t_o} \qquad x^* = \frac{x}{x_o} = \frac{x}{L_o}$$

$$\frac{d}{dt}(U_o U^*) = A\frac{d^2}{dx^2}(U_o U^*) \qquad \frac{d}{dt} = \frac{1}{t_o}\frac{d}{dt^*} \qquad \frac{d^2}{dx^2} = \frac{1}{L_o^2}\frac{d^2}{dx^{*2}}$$

$$\frac{U_o}{t_o}\frac{dU^*}{dt^*} = \frac{A\,U_o}{L_o^2}\frac{d^2U^*}{dx^{*2}} \Rightarrow \boxed{\frac{dU^*}{dt^*} = \frac{A\,t_o}{L_o^2}\frac{d^2U^*}{dx^{*2}}}$$

Review of Vector Calculus

$A = A_x \hat{i} + B_y \hat{j} + A_z \hat{k}$

$B = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

Dot Product (inner product)

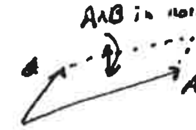
$$A \cdot B = A_x B_x + A_y B_y + A_z B_z = AB \cos\theta$$

Cross Product

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} - (A_x B_z - A_z B_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

= Area of parallelogram of sides A and B

A×B in normal direction



Derivatives of vectors

$$\frac{dA}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

also

$$\left.\begin{aligned} \frac{d}{dt}(aA) &= \frac{da}{dt}A + a\frac{dA}{dt} \\ \frac{d}{dt}(A \cdot B) &= A \cdot \frac{dB}{dt} + \frac{dA}{dt} \cdot B \\ \frac{d}{dt}(A \times B) &= A \times \frac{dB}{dt} + \frac{dA}{dt} \times B \end{aligned}\right\} \text{Same algebraic operations as scalars}$$

Field

$$F = f(x, y, z) \quad \text{or} \quad \text{Temp} = T(x, y, z)$$

Gradient ("del")

$$\nabla \phi = \text{grad } \phi$$

- in 3D cartesian $\nabla \phi = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}$
- gives direction vector of largest magnitude increase (slope)
- perpendicular to surface of constant ϕ

Directional Derivative

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{s} \qquad \text{where } \hat{s} \text{ is a unit vector}$$

Vector Calc (continued)

Divergence

$$\nabla \cdot A = \text{div}\, A = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

if $\nabla \cdot A = 0$, A is Solenoidal

Curl

$$\nabla \times A = \text{curl}\, A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix}$$

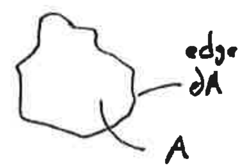
if $\nabla \times V = 0$, V is irrotational

Laplacian

$$\nabla^2 A = \nabla \cdot \nabla A = \text{div grad}\, A = \frac{d^2 A}{dx^2} + \frac{d^2 A}{dy^2} + \frac{d^2 A}{dz^2}$$

Green's theorem

$$\iint_A \left(\frac{\partial Q}{dx} - \frac{dP}{dy} \right) dx\, dy = \oint_{\partial A} P\, dx + Q\, dy$$



Divergence theorem

$$\iiint_A \text{div} V\, dA = \iint_{\partial A} V \cdot n\, d\hat{A}$$

Stokes theorem

$$\iint_\sigma (\text{curl}\, V) \cdot n\, d\sigma = \int_{\partial \sigma} V \cdot dr$$

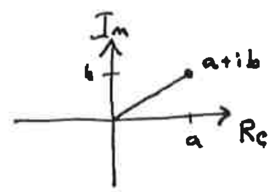
Divergence of curl

$$\nabla \cdot (\nabla \times V) = \text{div}(\text{curl}\, V) = 0$$

Curl of curl

$$\nabla \times (\nabla \times V) = \nabla(\nabla \cdot V) - \nabla^2 V$$

Review of Complex #s



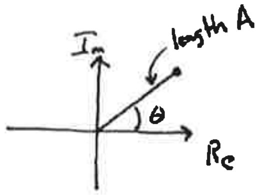
Number with Real and Imaginary Components

$$z = a + ib$$

polar form

$$Z = A e^{i\theta}$$

modulus ↑ argument ↑



Conversion

$$A = \sqrt{a^2+b^2} = |z| \qquad \theta = \operatorname{atan}\frac{y}{x}$$

Conjugate

$$\bar{z} = x - iy = re^{-i\theta} \quad \text{when} \quad z = x+iy = re^{i\theta}$$

identities:

$$|z| = |\bar{z}| , \quad \arg \bar{z} = -\arg z , \quad |z| = \sqrt{z\bar{z}}$$

$$\bar{\bar{z}} = z , \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

Operations

Add $z_1+z_2 = x_1+iy_1 + x_2+iy_2 = (x_1+x_2) + i(y_1+y_2)$

Mult $z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

power $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$

roots $z^{1/n} = r^{1/n} e^{i\theta/n}$

ex: $\sqrt{-1}$

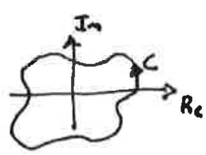
$\pi/2 \Rightarrow \sqrt{-1} = \underbrace{\pm i}_{\text{2 of them}}$

$\sqrt[4]{-1}$

$\pi/4 \quad \sqrt[4]{-1} = \underbrace{\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}}_{\text{4 of them}}$

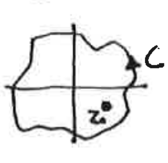
Cauchy Integral formula

- $f(z)$ analytic everywhere $\Rightarrow \int_C f(z)\,dz = 0$



Analytic ≡ derivative exists at a point

- $f(z)$ analytic except at z_k inside C



$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{\infty} \text{Residue}(z_k)$$

refer to a complex variable source for why.

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw} \qquad \text{Residue}(z_k) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

ODE

$$y' + y = 0 \qquad y(0) = A$$

$$y' + y = e^x$$

$$\underbrace{y' + y}_{} = 1 \qquad y(0) = 1$$

$$Ae^{\alpha x} + B + C\left(e^{-\alpha x}\right) + D\sin\left(x^{15}\right)$$

$$\left.\begin{aligned} y'' + y &= 0 \\ y'' - y &= 0 \end{aligned}\right\} \quad \begin{matrix} e^{ax} & e^{-ax} \\ \sin(x) & \cos(x) \end{matrix}$$

$$y'' + ay' + by = f$$