

GES 554

Lesson 2

15<sup>th</sup> Jan 16

Upcoming:

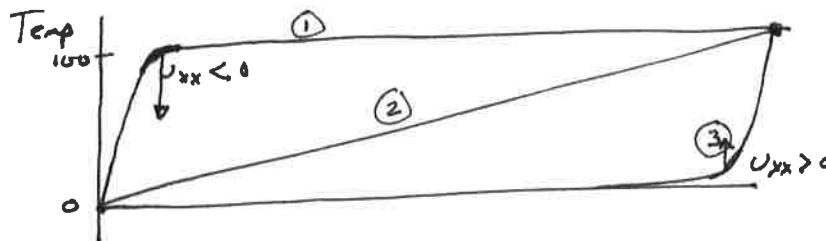
Read Farlow Ch. 1-5

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# Part I: Diffusion

## Diffusion Examples

- Heat flow in a bar

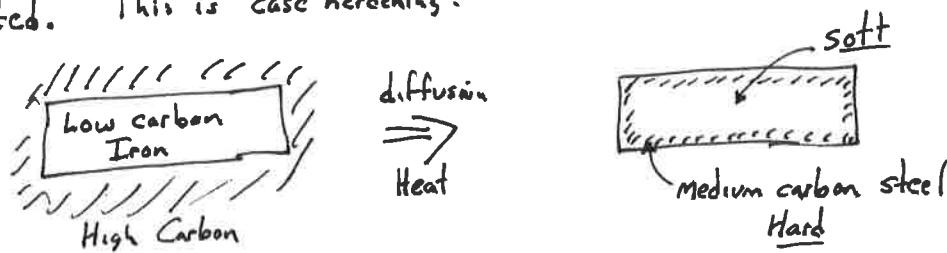


Show  $\frac{dU}{dx}$  behavior

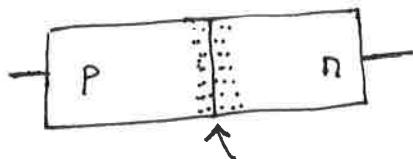
$$U_t = \alpha^2 U_{xx}$$

- Carbon in steel

Long ago, low carbon iron was surrounded by high carbon materials (hoof + horn!) and heated. This is case hardening.



- Semiconductors



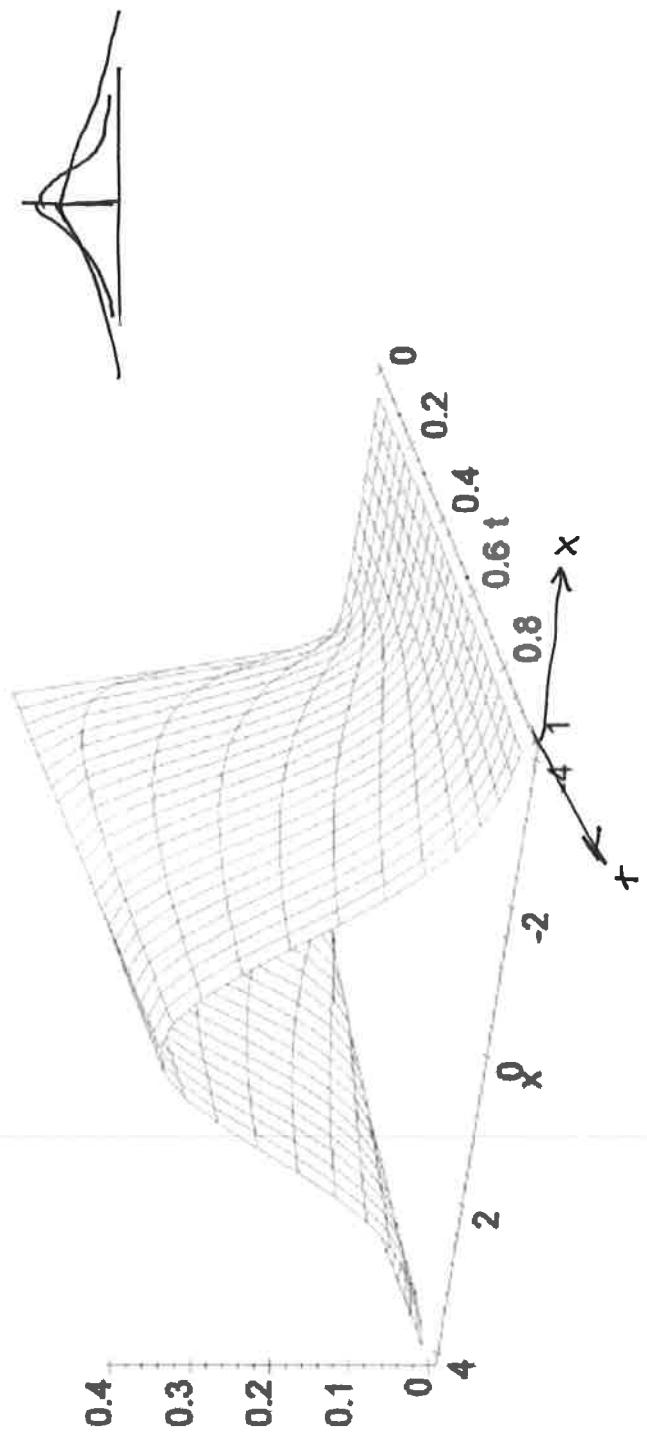
holes and electrons "drift" with applied voltage  
When put together, we can make a switch.

- Fluid Dynamics

"Tracing particles is too hard, so we will pretend that convection behaves like diffusion in the macro scale"

Viscosity

## Fundamental Solution (Salsa)



Show Video of  
Cooling Metal

<http://tiny.cc/GES554-hotball>

Discussion of Red Hot Nickel Ball being cooled in water:

- What are the relevant physics?
- What are the boundary conditions?
  - Is liquid water actually in contact with the ball?
  - Leidenfrost effect?
- Is the cooling symmetrical?
- What accounts for the difference in sound spectra as the cooling progresses?
  - Why nickel and not steel?

Non-technical issues:

- Engineers, could the marketplace have demand for this?
- Business opportunity?

## Diffusion Equation (Parabolic Equations)

- We will initially look at the diffusion equations in terms of heat transfer although the equations model more than heat transfer.

$$u_t - Du_{xx} = f(x, t)$$

- Change in scalar state is a linear combination of the state's curvature multiplied by a diffusion constant and a forcing term.
- Steady state with no forcing gives what function?
- Boundary conditions
  - Initial condition for the state
  - Field's edge states

## Derivation of the Heat Equation

- What are we really measuring? Temperature is an integrated effect.  
The raw states are atoms and their motion (momentum).
- Zoom into the vibrating atomic level of the substance. For a solid, the atoms are essentially vibrating in place. The scale of vibration determines the temperature.
- So we can reason that if a gradient of vibration scale exists, there will be a transfer of momentum in the negative gradient direction.
- Transferring this concept to temperature and heat flux, the heat flux at a point (for the solid) is likely to be proportional to the temperature gradient.

$$q \propto -\nabla T$$

- For a strict study of how temperature and atomic vibration are related, a course in rarified flows (microflows) would be in order.

Can we prove that a solution to the diffusion equation is unique?

- Let there be two independent solutions called  $u(x,t)$  and  $v(x,t)$
- Define a difference between these two solutions

$$w(x,t) = u(x,t) - v(x,t)$$

- This new solution still satisfies the governing equation. Why? Linear.
- $w_t - Dw_{xx} = 0$
- Furthermore, the boundary and initial conditions are still satisfied

$$w(0,t) = w(L,t) = 0$$

- Now we perform an operation that is extremely common in PDE theory and numerical formulation. Multiply the solution by a function and integrate over the domain.

$$\int_{\Omega} w w_t d\Omega - D \int_{\Omega} w w_{xx} d\Omega = 0$$

- With a sufficiently smooth (!!) solution, we can swap the integral and derivative order on the temporal term *Looks like energy!*

$$\int_{\Omega} w w_t = \frac{1}{2} \int_{\Omega} \frac{d(w^2)}{dt} d\Omega = \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 d\Omega$$

- On the other, we integrate by parts (Green's Identity)
- $$-D \int_{\Omega} w w_{xx} d\Omega = -D \int_{\partial\Omega} w w_x d\Omega + D \int_{\Omega} (w_x)^2 d\Omega$$

- On the boundary, w is zero. The first term is always zero. Internally, we have a positive function (and D is positive).

$$-D \int_{\Omega} w w_{xx} d\Omega \geq 0$$

- Recombining ~~w~~ gives

$$\frac{dE}{dt} \leq 0$$

- But since the initial energy is zero ( $w=0$  at  $t=0$ ), the u and v solutions are always identical for all time. Unique Soln Exists!

What happens if we run the Diffusion equation backwards in time?

What happens if the diffusion constant, D, is negative?

Perform the previous uniqueness proof to find

$$\frac{dE}{dt} \geq 0$$

Does every PDE have a solution? No.

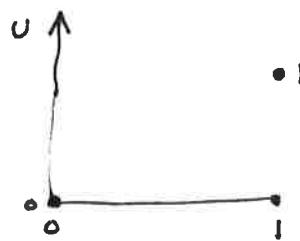
- You can imagine a non-linear PDE with horrid behavior  $u_x^2 = \frac{1}{1-t}$
- Q) Does every linear, constant coefficient PDE have a solution?
- Trivial Example  $u_x = 0$  with  $u(0) \neq u(L)$

## Constant Internal Heat Flow

$$U_t = \alpha^2 U_{xx} + 1 \quad 0 < x < 1$$

$$U(0,t) = 0$$

$$U(1,t) = 1$$



What is the steady state temp?

$$U_t = 0 \rightarrow \alpha^2 U_{xx} + 1 = 0 \rightarrow U_{xx} = C = -\frac{1}{\alpha^2}$$

Integrate

$$U_x = Cx + A$$

Integrate

$$U = \frac{Cx^2}{2} + Ax + B$$

BCs.

$$U(0) = 0 = C \cdot 0 + A \cdot 0 + B \rightarrow B = 0$$

$$U(1) = 1 = \frac{1}{2}C + A \rightarrow A = 1 - \frac{1}{2}C$$

Subst.

$$U = \frac{Cx^2}{2} + \left(1 - \frac{1}{2}C\right)x$$

$$U = -\frac{1}{\alpha^2} \frac{1}{2}x^2 + \left(1 + \frac{1}{2}\alpha^2\right)x$$

or better

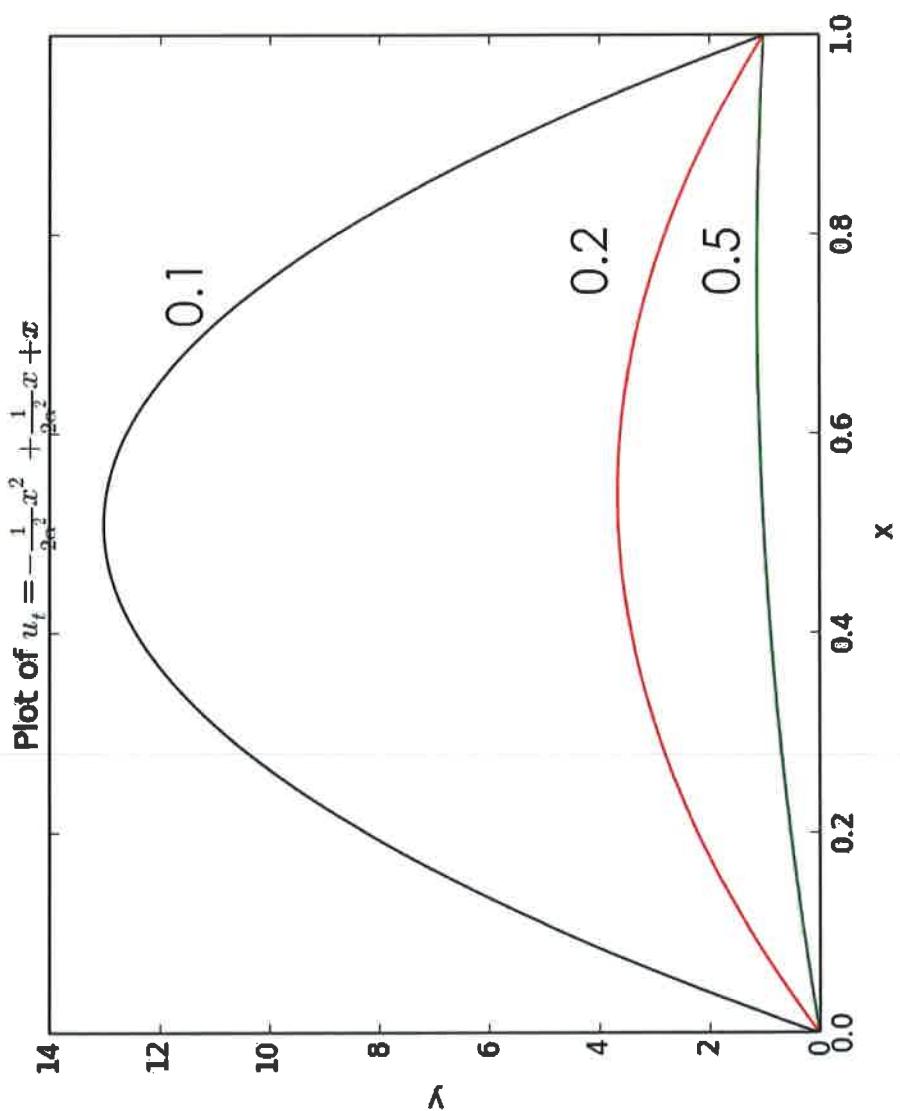
$$U = -\frac{1}{2\alpha^2}x^2 + \frac{1}{2\alpha^2}x + X$$

why is this better?

$U = x$  is homogeneous solution

Computer plot

$$\begin{aligned}u_t &= \alpha u_{xx} + 1 \\u(0, t) &= 0 \\u(1, t) &= 1\end{aligned}$$



Redux : RHN B

$$\begin{array}{c} S \\ \oplus \\ \hline C \\ \vdots \\ K \\ \vdots \\ P \end{array}$$

What is the governing equation for the temp?

$$U_t - k \nabla^2 U = 0$$

This is a spherical domain

$$\nabla^2 U = U_{rr} + \frac{2}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + \frac{\cot\theta}{r^2} U_\theta + \frac{1}{r^2 \sin\theta} U_{\phi\phi}$$

Symmetry  $\phi$  and  $\theta$  are not ~~important~~ (no orientation)

$$\nabla^2 U = U_{rr} + \frac{2}{r} U_r$$

Gov Eqs.

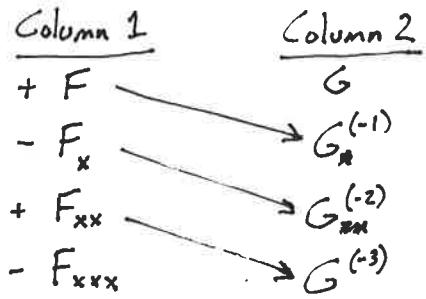
$$\boxed{U_t - k U_{rr} - \frac{2k}{r} U_r = 0}$$

- What possible problems do you see?
- What are BCs?  $U(r=0) = ?$   
 $U_r(r=0) = ?$

Can we solve this yet? Almost.

# Tabular Integration by Parts

$$\int F(x) G(x) dx$$



$$\int F(x) G(x) dx = \sum_{i=1}^N F^{(i-1)} G^{(-i)} + \int F^{(N-1)} G^{(-N)} dx$$

Example

$$\int x^2 \sin x dx = -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x)$$

<u>Col 1</u>	<u>Col 2</u>
$x^2$	$\sin x$
$-2x$	$-\cos(x)$
$2$	$-\sin(x)$
$0$	$\cos(x)$
$0$	...

$$\begin{aligned} \int_a^b x^2 \sin x dx &= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) \\ &= -b^2 \cos(b) + 2b \sin(b) + 2 \cos(b) \\ &\quad - (-a^2 \cos(a) + 2a \sin(a) + 2 \cos(a)) \end{aligned}$$

$$\int x^3 \cos x dx$$

$x^3$	$\cos x$	$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x$
$-3x^2$	$\sin x$	$+ 0$
$6x$	$-\cos x$	
$-6$	$-\sin x$	
$0$	$\cos x$	
	$\sin x$	