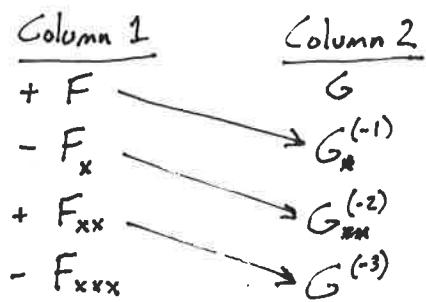


GES 554

Lecture 3

# Tabular Integration by Parts

$$\int F(x) G(x) dx$$



$$\int F(x) G(x) dx = \sum_{i=1}^N F^{(i-1)} G^{(-i)} + \int F^{(N-1)} G^{(-N)} dx$$

Example

$$\int x^2 \sin x dx = -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x)$$

| <u>Col 1</u> | <u>Col 2</u> |  |
|--------------|--------------|--|
| $x^2$        | $\sin x$     |  |
| $-2x$        | $-\cos(x)$   |  |
| $2$          | $-\sin(x)$   |  |
| $0$          | $\cos(x)$    |  |
| $0$          | $\dots$      |  |

$$\begin{aligned} \int_a^b x^2 \sin x dx &= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) \\ &= -b^2 \cos(b) + 2b \sin(b) + 2 \cos(b) \\ &\quad - (-a^2 \cos(a) + 2a \sin(a) + 2 \cos(a)) \end{aligned}$$

$$\int x^3 \cos x dx$$

|         |           |   |
|---------|-----------|---|
| $x^3$   | $\cos x$  | $= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6 \cos x$ |
| $-3x^2$ | $\sin x$  | $+ 0$   |
| $6x$    | $-\cos x$ |   |
| $-6$    | $-\sin x$ |   |
| $0$     | $\cos x$  |   |
|         | $\sin x$  |   |

# L2P3

$$U_t = \alpha^2 U_{xx} - \beta U \quad U(0,t) = 1 \quad U(1,t) = 1$$

Find S. State.

$$U_t = 0 \Rightarrow \alpha^2 U_{xx} - \beta U = 0 \Rightarrow U_{xx} - \frac{\beta}{\alpha^2} U = 0$$

Prospective soln

$$U = A e^{cx} + B e^{-cx}$$

$$U_{xx} - c^2 U = 0$$

Test

$$U_{xx} = A c^2 e^{cx} + B c^2 e^{-cx}$$

$$A c^2 e^{cx} + B c^2 e^{-cx} - c^2 A c^{cx} - c^2 B e^{-cx} \stackrel{?}{=} 0 \quad \text{true } \checkmark$$

Fit BCs.

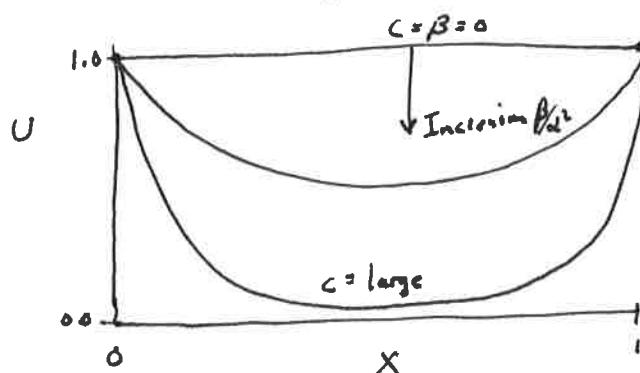
$$\begin{aligned} U(0) = 1 &= A e^{c \cdot 0} + B e^{-c \cdot 0} = A + B \Rightarrow A = 1 - B \\ U(1) = 1 &= A e^c + B e^{-c} = A e^c + B e^{-c} \end{aligned}$$

$$U = (1 - B) e^{cx} + B e^{-cx} = 1 \Rightarrow e^c - B(e^c - e^{-c}) = 1$$

$$B = \frac{e^c - 1}{e^c - e^{-c}}$$

Soln

$$U = \left(1 - \frac{e^c - 1}{e^c - e^{-c}}\right) e^{cx} + \left(\frac{e^c - 1}{e^c - e^{-c}}\right) e^{-cx}$$



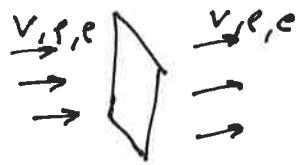
- Physically, what is happening?

- Can U < 0?

Flux:

How much  $v$  passes a given area per amount of time.

$$F_{\text{mass}} = \rho v$$



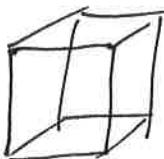
$$F_{\text{momentum}} = v(v_p) = \rho v^2$$

$$F_{\text{energy}} = v e p$$

## Transport Equation

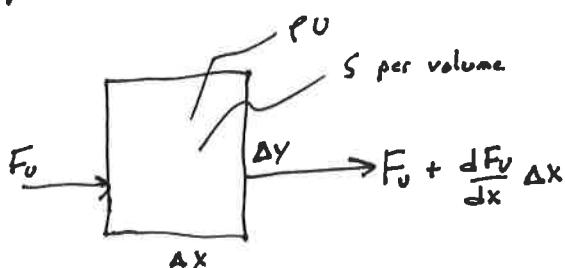
Change in state per given time = Amount Entering per given time - Amount Exiting per given time

~~Area~~ ~~ΔP ΔA Δt~~ ~~ΔV~~ ~~ΔS~~ ~~ΔF~~



$$\frac{du}{dt} + \iint_S F_u \cdot dS = S$$
$$\frac{du}{dt} + \nabla \cdot F = S$$

Derivative.

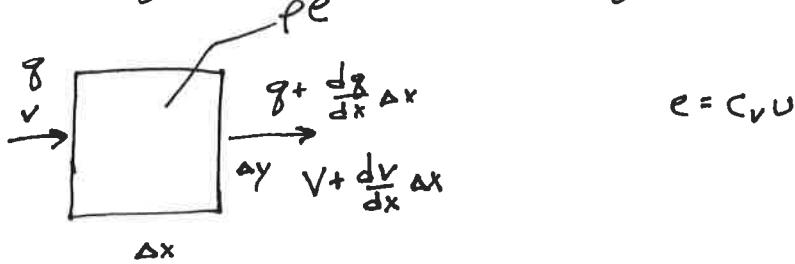


$$\Delta x \Delta y \frac{d(\rho u)}{dt} = \left( F_u - \cancel{F_u} - \frac{dF_u}{dx} \Delta x \right) \Delta y + S \Delta x \Delta y$$

$$\boxed{\frac{d(\rho u)}{dt} + \frac{dF_u}{dx} = S}$$

FL4PY

4. Add a moving stream to Diffusion eqn.



$$e = C_v u$$

Continuity of Mass.

$$\begin{aligned} \frac{d}{dt}(\rho \Delta x \Delta y) &= \rho v \Delta y - \cancel{\rho} \left( v + \frac{dv}{dx} \Delta x \right) \Delta y \\ &= \cancel{\rho} v \Delta y - \cancel{\rho} v \Delta y - \rho \frac{dv}{dx} \Delta x \Delta y - v \frac{dp}{dx} \Delta x \Delta y \end{aligned}$$

$$\frac{dp}{dt} = -\rho \frac{dv}{dx} - v \frac{dp}{dx}$$

Do we really need this?  
If  $\rho$  is constant ( $H_2O$ )  
then  $\frac{dp}{dx} = 0$

Continuity of Energy

Direct shortcut to Transport equation  $\bar{w} F_{\text{Diff}} = -K \nabla T$  and  $F_{\text{cond}} = v \rho e$ 

$$\text{so } \frac{d(\rho e)}{dt} + \frac{1}{\rho} \left( -K \nabla^2 T + v \rho e \right) = 0$$

$$\frac{C_v \rho}{\rho} \frac{d\rho e}{dt} + -K \nabla^2 T + v \frac{1}{\rho} \left( \frac{\rho e}{C_v} \right) = 0$$

$$\boxed{\frac{du}{dt} - \frac{K}{C_v \rho} \nabla^2 T + v \frac{1}{\rho} \frac{d}{dx}(u) = 0}$$

## Separation of Variables (Lesson 5)

- Applies to linear homogeneous PDEs ~~to~~

$$U_t = cL(U, U_x, U_{xx})$$

$L$  is a linear operator  
(can include derivatives)

- Assume solution is

$$\text{over } X(t) T(x)$$

$$U = X(x) T(t)$$

- Subs

$$\frac{d}{dt}(X(x) T(t)) = c L(X(x) T(t))$$

- $X$  and  $T$  are only functions of  $x$  and  $t$  resp.

$$X \frac{dT(t)}{dt} = c T(t) L(X(x))$$

- Divide by  $X(t) T(t)$

$$\frac{1}{T(t)} \frac{dT(t)}{dt} = \frac{c L(X(x))}{X(x)}$$

- Only  $t$  and only  $x$ , the LHS and RHS are completely independent, so each must be equal to a constant.

$$c \frac{T_t}{T} = \cancel{c} \frac{L(X)}{X} = K$$

- Amazing, now we have two independent ODEs!

$$\boxed{T_t = cK T \quad \text{and} \quad L(X) = K X}$$

$$L = U_{xx}$$

- For a canonical derivation, pick  $K = -\lambda^2$  so diffusion eqn solution is

$$\boxed{T(t) = \cancel{T_0} e^{-\lambda^2 c t} \quad X(x) = A \sin(\lambda x) + B \cos(\lambda x) \xrightarrow{\text{BC}}}$$

$\lambda$  fits BC.

Computer plots

# Diffusion Equation Separation of Variables

$$U_t = K \frac{d^2 U}{dx^2}$$

Sep of Vars

$$U = X(x) T(t)$$

Subs

$$\frac{d}{dt} (X(x) T(t)) = K T(t) \frac{d^2 X(x)}{dx^2}$$

Collect terms.

$$\frac{1}{K T} \frac{dT(t)}{dt} = \frac{d^2 X(x)}{dx^2} \frac{1}{X(x)} = -\lambda^2$$

$x$  and  $t$  are independent

$$T_t = -K T \lambda^2 \Rightarrow T(t) = T_0 e^{-K \lambda_i^2 t}$$

$$X_{xx} = -\lambda^2 X \quad \text{this is an eigenfunction (SL) problem.}$$

$$\Rightarrow X(x) = \sum_{i=1}^N A_i \sin(\lambda_i x) \quad \text{for } \lambda \text{ fits BCs}$$

Solution

$$U = T_0 e^{-K \lambda_i^2 t} A_i \sin(\lambda_i x)$$

• Sep Vars

$$\lambda^2$$

Now, you have all of the tools necessary to solve the generic homogeneous Diffusion eqn.

- 1) Compute Fourier terms to represent Initial Conditions
- 2) Solve Eigenvalue Problem for  $X(x)$
- 3) Compute  $U = T(t) \cdot X(x)$

L5P1 Show  $u(x,t) = e^{-\lambda^2 \alpha^2 t} [A \sin(\lambda x) + B \cos(\lambda x)]$  satisfies the PDE  $u_t = \alpha^2 u_{xx}$

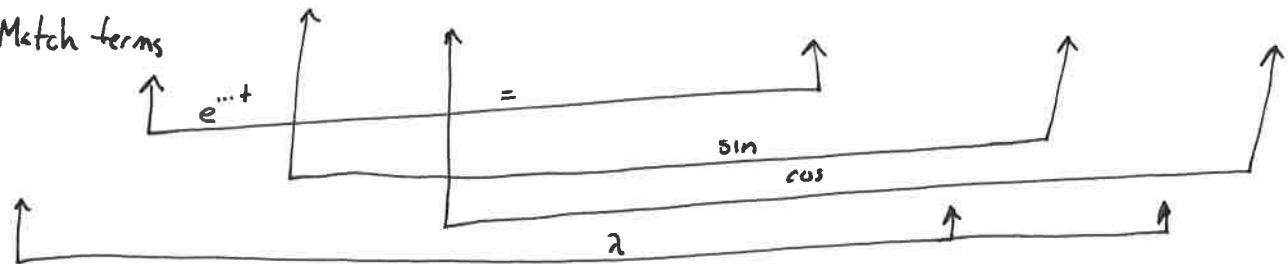
$$u_t = -\lambda^2 \alpha^2 e^{-\lambda^2 \alpha^2 t} [A \sin(\lambda x) + B \cos(\lambda x)]$$

$$u_{xx} = e^{-\lambda^2 \alpha^2 t} [-A \lambda^2 \sin(\lambda x) - \lambda^2 B \cos(\lambda x)]$$

Substitute:  $\overset{\uparrow}{u_t} = \alpha^2 \overset{\uparrow}{u_{xx}}$

$$\underset{\alpha^2}{-\lambda^2 \alpha^2 \left[ e^{-\lambda^2 \alpha^2 t} \right] \left[ A \sin(\lambda x) + B \cos(\lambda x) \right]} = \alpha^2 \left[ e^{-\lambda^2 \alpha^2 t} \right] \left[ -A \lambda^2 \sin(\lambda x) - \lambda^2 B \cos(\lambda x) \right]$$

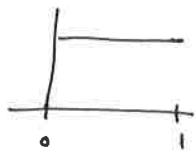
Match terms



Equal



L5P3

Find the Fourier sine expansion of  $\phi(x) = 1$ 

$$\phi(x) = A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \dots$$

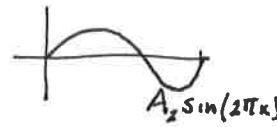
$$\begin{aligned}
 A_m &= 2 \int_0^1 \phi(x) \sin(m\pi x) dx = 2 \int_0^1 1 \cdot \sin(m\pi x) dx = -\frac{1}{m\pi} \cos(m\pi x) \Big|_0^1 \\
 &= -\frac{1}{m\pi} [\cos(m\pi) - \cos 0] \\
 &= -\frac{1}{m\pi} [(-1)^m - 1]
 \end{aligned}$$

$$A_1 = +\frac{2}{\pi}$$

$$A_2 = -\frac{2}{2\pi} \cdot 0 = 0 \quad \leftarrow \text{we should expect this to be zero! Why?}$$

$$A_3 = \frac{2}{3\pi}$$

$$A_4 = 0$$



but

$$\int f \cdot F = \int f$$

and the integral of  
is zero!

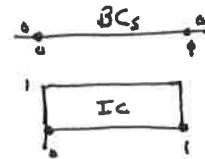
Verify in Farlow Table D # 6.

$$f(x) = 1 \Rightarrow S_n = \frac{2}{n\pi} [1 - (-1)^n]$$



L5P4

Solution to  $U_t = U_{xx}$        $0 < x < 1$   
 $U(0,t) = 0$        $0 < t < \infty$   
 $U(1,t) = 0$   
IC  $U(x,0) = 1$        $0 < x < 1$



• SoV:

$$U = X(x) T(t) \Rightarrow U_t = X(x) T'_t(t) \quad \text{and} \quad U_{xx} = X_{xx}(x) T(t)$$

• Substitute into gov eqn:

$$X(x) T'_t(t) = X_{xx}(x) T(t)$$

• Pull all x and t terms to separate sides

$$\frac{T'_t}{T} = \frac{X_{xx}}{X}$$

• Since these are equal always and there is no interdependency between x and t

$$\frac{T'_t}{T} = \frac{X_{xx}}{X} = -\lambda^2$$

arbitrary constant picked for later convenience.

• X term equation

$$\frac{X_{xx}}{X} = -\lambda^2 \Rightarrow \underline{\underline{X_{xx} + \lambda^2 X = 0}}$$

• T term equation

$$\frac{T'_t}{T} = -\lambda^2 \Rightarrow \underline{\underline{T'_t + \lambda^2 T = 0}}$$

- $T$  term solution to  $T_+ + \lambda^2 T = 0$  (linear 1<sup>st</sup> order)

Expect a solution of the form  $T(t) = A e^{-\lambda^2 t}$

The initial condition is  $u(x, 0) = 1 \Rightarrow 1 = X(x) T(0)$

$$= \underbrace{X(x)}_{\text{Leave the IC scaling with } X \text{ not } T} A e^{-\lambda^2 0}$$

Leave the IC scaling with  $X$  not  $T$ .

- $X$  term solution to  $X_{xx} + \lambda^2 X = 0$  with  $u(0, t) = 0$   
 $u(1, t) = 0$

Expect  $X(x) = A \sin(\lambda x) + B \cos(\lambda x)$

Apply BCs

$$u(0, t) = 0 \Rightarrow \underbrace{X(0) T(t)}_{\substack{\text{If } T \text{ is arbitrary} \\ \text{then } X \text{ must be zero always}}} \Rightarrow X(0) = 0 = A \sin(0) + B \cos(0)$$

Thus  $B = 0$

$$u(1, t) = 0 \Rightarrow X(1) = 0 = A \sin(\lambda) + \cancel{B \cos(\lambda)}$$

This is an eigenvalue problem. Only certain values of  $\lambda$  are possible. Theory soon.

$0 = A \sin(\lambda)$  either  $A = 0$  boring solution!!

or  $\sin \lambda = 0 \Rightarrow \lambda = n\pi$

$X$  solution

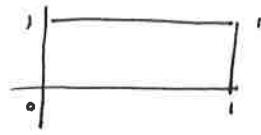


$$X(x) = A \sin(n\pi x)$$

But what is the value of  $A$ ?

## Initial Conditions

$$U(x, 0) = 1 = X(x) T(0)^{\cancel{1}} \Rightarrow X(x) = 1$$



$$X(x) = 1 = A_n \underbrace{\sin(n\pi x)}_{}$$

Write as a Fourier Series of  $X(x)$

The series is

$$1 = A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + A_4 \sin(4\pi x) + \dots$$

How can we find values for an infinite number of  $A_m$  coefficients with only one equation?

The  $X(x)$  terms ( $\sin(n\pi x)$ ) are orthogonal and the  $A_m$  terms can be determined piece by piece.

$$\int_0^1 A_m \sin(n\pi x) \cdot \sin(m\pi x) dx = \int_0^1 1 \cdot \sin(m\pi x) dx$$

$$= \begin{cases} \frac{A_m}{2} & \text{when } n=m \\ 0 & \text{otherwise} \end{cases}$$

$$-\frac{1}{m\pi} \cos(m\pi x) \Big|_0^1 = -\frac{1}{m\pi} [(-1)^m - 1]$$

Notice that this is given generically as

$$A_m = 2 \int_0^1 \phi(x) \sin(n\pi x) dx \quad (\text{Eq 5.5 on page 39})$$

$$A_1 = \frac{4}{\pi}, A_2 = 0, A_3 = \cancel{\frac{4}{3\pi}}, A_4 = 0, A_5 = \frac{4}{5\pi}$$

## Complete Solution

$$U = X(x) T(t) = A_n \underbrace{\sin(n\pi x)}_{\tilde{a}} e^{-\lambda^2 t} = A_n \sin(n\pi x) e^{-n^2 \pi^2 t} = A_1 \sin(\pi x) e^{-\pi^2 t} + A_2 \sin(2\pi x) e^{-4\pi^2 t} + A_3 \sin(3\pi x) e^{-9\pi^2 t} + \dots$$

$$U(x, t) = \frac{4}{\pi} \sin(\pi x) e^{-\pi^2 t} + \frac{4}{3\pi} \sin(2\pi x) e^{-4\pi^2 t} + \dots + -\frac{2}{m\pi} [(-1)^m - 1] \sin(m\pi x) e^{-m^2 \pi^2 t}$$

## Discussion:

- Solution decays to zero as  $t \rightarrow \infty$

$$T(t) = T_0 e^{-n^2 \pi^2 t}$$

- Higher frequency terms decay with  $n^2$

"Diffusion relentlessly eliminates noise" 

- For mid-times, the sole remaining term is  $\frac{4}{\pi} \sin(\pi x) e^{-\pi^2 t}$



- The solution is symmetric about  $x=0.5$ . In other words  $A_{\text{even}} = 0$  Always

- A truncated series (i.e.  $A_0 \dots A_N$  where  $N < \infty$ ) gives ~~Gibbs~~ ringing at the step change in ICs. (at  $x=0$  and  $x=1$ )



This numerical error is not significant since these high frequency terms are squelched by diffusion.

L5P5

$$U(x,0) = \sin(2\pi x) + \frac{1}{3} \sin(4\pi x) + \frac{1}{5} \sin(6\pi x)$$

This is an easy problem since the initial conditions are already in a Fourier series!

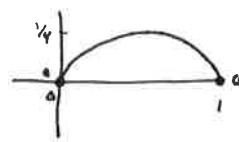
You can directly read off the coefficients and write the solution on one line.

$$U(x,t) = \underbrace{1 \cdot \sin(2\pi x)}_{\substack{\text{coeff} \\ \text{of} \\ \text{1st} \\ \text{term}}} \underbrace{e^{-4\pi^2 t}}_{\substack{\text{1st term} \\ \text{at IC}}} + \underbrace{\frac{1}{3} \cdot \sin(4\pi x)}_{\substack{\text{time} \\ \text{decay} \\ \text{of} \\ \text{1st term}}} \underbrace{e^{-16\pi^2 t}}_{\substack{A_4 \\ X_4 \\ T_4}} + \underbrace{\frac{1}{5} \cdot \sin(6\pi x)}_{\substack{A_6 \\ X_6 \\ T_6}} \underbrace{e^{-36\pi^2 t}}_{\substack{-36\pi^2 t}}$$



LSP6

$$\text{IC is } u(x,0) = x - x^2$$



Find a Fourier expansion of  $u(x,0)$

$$\text{We know that } A_m = 2 \int_0^1 (x - x^2) \sin(m\pi x) dx$$

I will use a tabular Integration by parts approach.

| sign | table 1               | table 2  |
|------|-----------------------|--|
| +    | $\rightarrow x - x^2$ | $\sin(m\pi x)$                                 |
| -    | $\rightarrow 1 - 2x$  | $\rightarrow -\frac{1}{m\pi} \cos(m\pi x)$     |
| +    | $\rightarrow -2$      | $\rightarrow -\frac{1}{m^2\pi^2} \sin(m\pi x)$ |
| -    | $\rightarrow 0$       | $\rightarrow +\frac{1}{m^3\pi^3} \cos(m\pi x)$ |

$$= (x - x^2) \frac{-1}{m\pi} \cos(m\pi x) - (1 - 2x) \frac{-1}{m^2\pi^2} \sin(m\pi x) + (-2) \frac{1}{m^3\pi^3} \cos(m\pi x) \Big|_0^1$$

Expand

$$= (1 - 1) \left( \frac{-1}{m\pi} \right) \cos(m\pi) - (1 - 2) \left( \frac{-1}{m^2\pi^2} \right) \sin(m\pi) + (-2) \frac{1}{m^3\pi^3} \cos(m\pi) \\ - (1 - 1) \left( \frac{-1}{m\pi} \right) \cos(0) + \dots \sin(0) \neq (-2) \frac{1}{m^3\pi^3} \cos(0)$$

Cancel terms that are zero

$$= 0 - 0 + \frac{-2}{m^3\pi^3} \cos(m\pi) \\ - 0 + 0 - \frac{-2}{m^3\pi^3} 1$$

$$A_m = 2 \left( -\frac{2}{m^3\pi^3} \cos(m\pi) + \frac{2}{m^3\pi^3} \right) = \frac{4}{m^3\pi^3} \left( 1 - (-1)^{m+1} \right)$$

$$= \frac{4}{m^3\pi^3} \left( 1 + (-1)^m \right)$$

$$A_1 = \frac{8}{\pi^3} \quad A_2 = 0 \quad A_3 = \frac{8}{27\pi^3}, \quad A_5 = \frac{8}{125\pi^3}$$

↖ Again, we should  
know this by  
symmetry.

$$U(x,t) = \frac{8}{\pi^3} \sin(\pi x) e^{-\pi^2 t} + \frac{8}{27\pi^3} \sin(3\pi x) e^{-9\pi^2 t} + \frac{8}{125\pi^3} \sin(5\pi x) e^{-25\pi^2 t} + \dots$$

Q: When do we truncate the Fourier series?

A: When adding terms does not change our conclusion / analysis.

$$A_5 \approx 0.002 \quad \ll \quad A_1 \approx \underline{\underline{0.258}}$$

For this problem,  $x - x^2$  looks a lot like  $\frac{\sin(\pi x)}{4}$

$$U(x,t) \approx \frac{8}{\pi^3} \sin(\pi x) e^{-\pi^2 t}$$

$\approx 97\%$  of the solution in 1 term!

Not all problems are this way