

GES 554

Lecture 4

Boundary Conditions

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Non-Homogeneous BCs

Lesson 6

Boundary Conditions

- Dirichlet ("direct") ex: $U(0) = 0$
The value is specified
- Neumann ("derivatives") ex: $\frac{du}{dx}(x=0) = 1$
The derivative is specified
- Mixed ("combination of Dirichlet and Neumann") ex: $\frac{du}{dx}(x=0) + u(0) = 2$

- Special case is Robin BC.
Linear combination of value and derivative

$$\alpha U_x(BC) + \beta U(BC) = V$$

• Homogeneous BCs

"Zero"

Example: $U(0,t) + U_x(0,t) = 0$

• Non-Homogeneous BCs

"Non-Zero"

Example $U(0,t) + U_x(0,t) = 1$

To use separation of variables, we need the BCs in a homogeneous form.
zero

Unfortunately, PDEs are often non-homogeneous (Lesson 6)

$$\left. \begin{array}{l} U_t - \alpha^2 U_{xx} = f(x,t) \\ \alpha_1 U_x(0,t) + \beta_1 U(0,t) = g_1(t) \\ \alpha_2 U_x(L,t) + \beta_2 U(L,t) = g_2(t) \\ U(x,0) = \phi(x) \end{array} \right\} \begin{array}{l} \text{Governing Equation} \\ \text{Boundary Conditions} \\ \text{Initial Conditions} \end{array} \quad \left. \begin{array}{l} \text{parts} \\ \text{m} \end{array} \right\}$$

- Since the PDE is linear, superposition applies ($U = U_1 + U_2 + \dots$)

1) So, break the solution into two parts

- 1) steady state that fits BCs
- 2) transient that fits ICs

Does this need to be
"the" exact s.s. solution? NO!

$$U(x,t) = \cancel{\text{steady state}} \quad \bar{U} + U$$

$\nwarrow_{\text{s.s.}}$ $\nearrow_{\text{Transient}}$

2) Apply to Gov Egu, BCs, and IC.

$$\bar{U}_t + U_t - \alpha^2 (\bar{U}_{xx} + U_{xx}) = f(x,t)$$

$$\alpha_1 (\bar{U}_x(0,t) + U_x(0,t)) + \beta_1 (\bar{U}(0,t) + U(0,t)) = g_1(t)$$

...

$$\bar{U}(x,0) + U(x,0) = \phi(x)$$

3) Eliminate terms

$$\frac{d\bar{U}}{dt} = 0, \quad \text{BC terms at } \bar{U} \text{ cancel } g_1(t) \text{ and } g_2(t) !$$

why? You picked a good $\bar{U}(x)$!!! Very important.

4) Solve the resulting homogeneous BC PDE.

Warning: the Gov Egu and/or the IC equation may become more complicated.

What is the resulting PDE?

$$U_t - \alpha^2 (\bar{U}_{xx} + U_{xx}) = f(x,t)$$

pick a linear \bar{U} and this disappears
otherwise, move to RHS as a non-homogeneous
part with $f(x,t)$

$$\alpha_1 U_x(0,t) + \beta_1 U(0,t) = 0$$

$$\alpha_2 U_x(L,t) + \beta_2 U(L,t) = 0$$

$$U(x,0) = \phi(x) - \bar{U}(x,0)$$

↖ This method will complicate the IC.
But, this is easier to solve than
the original non-homogeneous BC!!

$$U = \bar{U} + U'$$

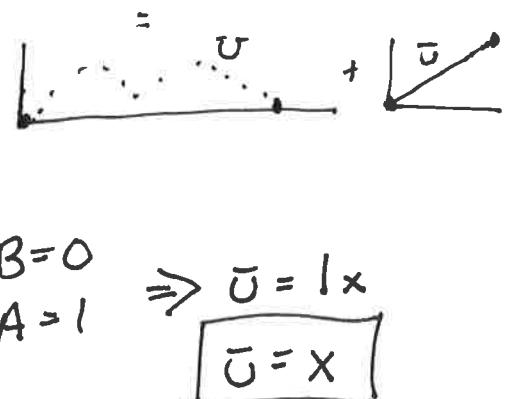
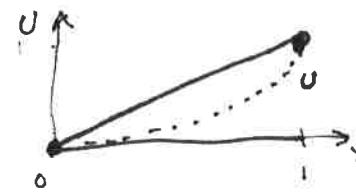
Lesson 6 Problem 2

$$U_t = U_{xx}$$

$$U(0,t) = 0$$

$$U(1,t) = 1$$

$$U(x,0) = x^2$$



- Find a steady state solution

$$\bar{U}_{xx} = 0 \Rightarrow \bar{U} = Ax + B$$

$$\begin{aligned} \text{Apply BCs: } \bar{U}(0) &= 0 = B \Rightarrow B = 0 \\ \bar{U}(1) &= A = 1 \Rightarrow A = 1 \end{aligned} \Rightarrow \boxed{\bar{U} = x}$$

- Apply $U = \bar{U} + U$ to Gov Egu, BCs, and ICs.

- GovEg: $\begin{aligned} U_t &= \bar{U}_t + U_t = 0 + U_t \\ U_{xx} &= \bar{U}_{xx} + U_{xx} = 0 + U_{xx} \end{aligned} \Rightarrow \boxed{U_t = U_{xx}}$

- BC₁: $U(0,t) = 0 = \bar{U}(0,t) + U(0,t) = 0 + U(0,t) \Rightarrow \boxed{U(0,t) = 0}$

- BC₂: $U(1,t) = 1 = \bar{U}(1,t) + U(1,t) = 1 + U(1,t) \Rightarrow \boxed{U(1,t) = 0}$

- IC: $U(x,0) = x^2 = \bar{U}(x,0) + U(x,0) = x + U(x,0) \Rightarrow \boxed{U(x,0) = x^2 - x}$

This is a nice clean PDE. (Homogeneous with sine series solution.)

- Separation of Variables $U = X\Gamma$

$$X\Gamma_t = X_{xx}\Gamma \implies \frac{\Gamma_t}{\Gamma} = \frac{X_{xx}}{X} = -\lambda^2 \quad X_{xx} + \lambda^2 X = 0$$

So we know $\Gamma(t) = e^{-\lambda^2 t}$

And $X = \sum A_m \sin(\pi x)$ where $A_m = 2 \int_0^1 \phi(x) \sin(m\pi x) dx$

$$A_m = 2 \int_0^1 (x^2 - x) \sin(m\pi x) dx$$

Integration By Parts (Tabular Method)

+	$x^2 - x$	$\sin(m\pi x)$
-	$2x - 1$	$-\cos(m\pi x) \frac{1}{m\pi}$
+	2	$-\sin(m\pi x) \frac{1}{m^2\pi^2}$
-	0	$\cos(m\pi x) \frac{1}{m^3\pi^3}$

...

$$\cancel{(x^2 - x)}^0 \cancel{(-\cos m\pi x) \frac{1}{m\pi}}^0 + \cancel{(2x - 1)}^1 \cancel{\sin(m\pi x) \frac{1}{m^2\pi^2}}^0 + 2 \cos m\pi x \frac{1}{m^3\pi^3} \Big|$$

$$\frac{2 \cos m\pi}{m^3\pi^3} - \frac{2 \cos 0}{m^3\pi^3}$$



$$A_{m_{\text{even}}} = 2 \left(\frac{2 \cos m\pi}{m^3\pi^3} - \frac{2}{m^3\pi^3} \right) = 0$$

$$A_{m_{\text{odd}}} = 2 \left(\frac{2 \cos m\pi}{m^3\pi^3} - \frac{2}{m^3\pi^3} \right) = 2 \left(\frac{-4}{m^3\pi^3} \right) = \underline{\underline{-\frac{8}{m^3\pi^3}}}$$

Complete Soln.

$$e^{-\lambda^2 t} \quad \lambda^2 = n^2$$

$$U = \bar{U} + u$$

$$U = x + \underbrace{-\frac{8}{\pi^3} e^{-\pi^2 t} \sin(\pi x)}_{m=1} + \underbrace{\frac{-8}{27\pi^3} e^{-9\pi^2 t} \sin(3\pi x)}_{m=3} + \dots$$

Is this the correct solution?

Gov Eqn.

$$U_t = -\frac{8}{\pi^3} -\pi^2 e^{-\pi^2 t} \sin \pi x + \dots$$

$$U_{xx} = -\frac{8}{\pi^3} e^{-\pi^2 t} (-\pi^2) \sin \pi x + \dots$$



BC

$$U(0, t) = 0 \quad \checkmark$$

BC

$$U(1, t) = 1 = 1 + () \sin(\pi)^t \quad \checkmark$$

IC

$$U(x, 0) = x^2 = x + \dots$$

This is not obvious.

Lesson 6 Problem 1.

$$U_t = \alpha^2 U_{xx}$$

$$U(0,+) = 1$$

$$U_x(1,+) + h U(1,+) = 1$$

$$U(x,0) = \sin(\pi x) + x$$

Soln:

- Transform into homogeneous BCs $U = \bar{U} + U$

- Find a steady state solution

$$\alpha^2 U_{xx} = 0 \Rightarrow U_{xx} = 0 \Rightarrow \bar{U} = Ax + B$$

Apply BCs $\bar{U}(0,+) = 1 = A \cdot 0 + B \Rightarrow B = 1$

$$\bar{U}_x(1,+) + h \bar{U}(1,+) = 1 \Rightarrow A + h(A \cdot 1 + B) = 1$$

$$A(1+h) = 1 - h$$

$$A = \left(\frac{1-h}{1+h} \right)$$

$$\bar{U} = \left(\frac{1-h}{1+h} \right)x + 1$$

- Apply $U = \bar{U} + U$ to Gov Eqs.

$$U_x = \bar{U}_x + U_x = \left(\frac{1-h}{1+h} \right) + U_x \quad \text{and } U_{xx} = \bar{U}_{xx} + U_{xx}$$

$$U_t = \cancel{\bar{U}_t} + U_t = U_t$$

$$\boxed{U_t = \alpha^2 U_{xx}} \text{ Gov Eq}$$

- Apply $U = \bar{U} + U$ to BC #1 ($x=0$)

$$\bar{U}(0,+) + U(0,+) = 1$$

$$\left(\frac{1-h}{1+h}\right)0 + 1 + U(0,+) = 1 \Rightarrow \boxed{U(0,+) = 0}$$

- Apply $U = \bar{U} + U$ to BC #2 ($x=1$)

$$\bar{U}_x(1,+) + U_x(1,+) + h\bar{U}(1,+) + hU(1,+) = 1$$

$$\left(\frac{1-h}{1+h}\right)\cancel{+} + U_x(1,+) + h\left(\frac{1-h}{1+h} + 1\right) + hU(1,+) = 1$$

$$U_x(1,+) + hU(1,+) = 1 - \frac{1-h}{1+h} - \frac{h-h^2}{1+h} - h$$

$$\boxed{U_x(1,+) + hU(1,+) = 0}$$

We now have a homogeneous ~~BC~~ BC.

- Apply $U = \bar{U} + U$ to ICs

$$\bar{U}(x,0) + U(x,0) = \sin(\pi x) + x$$

$$\left(\frac{1-h}{1+h}\right)x + 1 + U(x,0) = \sin(\pi x) + x$$

$$U(x,0) = \sin(\pi x) + x - 1 - \left(\frac{1-h}{1+h}\right)x$$

$$\boxed{U(x,0) = \sin(\pi x) + \underbrace{\frac{h^2}{1+h}x}_{-1}}$$

These terms are trouble!

The solution will have a series expansion.

• Separation of Variables

$$U = X(x) T(t)$$

Subst into Gov Egu

$$X(x) T_t(t) = \alpha^2 X_{xx}(x) T(t)$$

Collect x, t terms

$$\frac{1}{\alpha^2} \frac{T_t(t)}{T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda^2$$

Soln for $T(t)$

$$T_t(t) = -\lambda^2 \alpha^2 T(t) \Rightarrow T(t) = T_0 e^{-\lambda^2 \alpha^2 t}$$

Soln for $X(x)$

$$X_{xx}(x) = -\lambda^2 X(x)$$

solutions are

$$X(x) = A \sin(\lambda x) + B \cos(\lambda x)$$

But, what λ s satisfy the BCs?

Satisfy the BCs

- $U(0,+) = 0$

So, $X(0) = 0$

$$0 = A \sin(0) + B \cos(0)^1 \Rightarrow B = 0$$

- $U_x(1,+) + h U(1,+) = 0$

$$A\lambda \cos(\lambda) + h(A \sin \lambda) = 0$$

$$A(\lambda \cos \lambda + h \sin \lambda) = 0$$

So either $A=0$ (boring soln) or

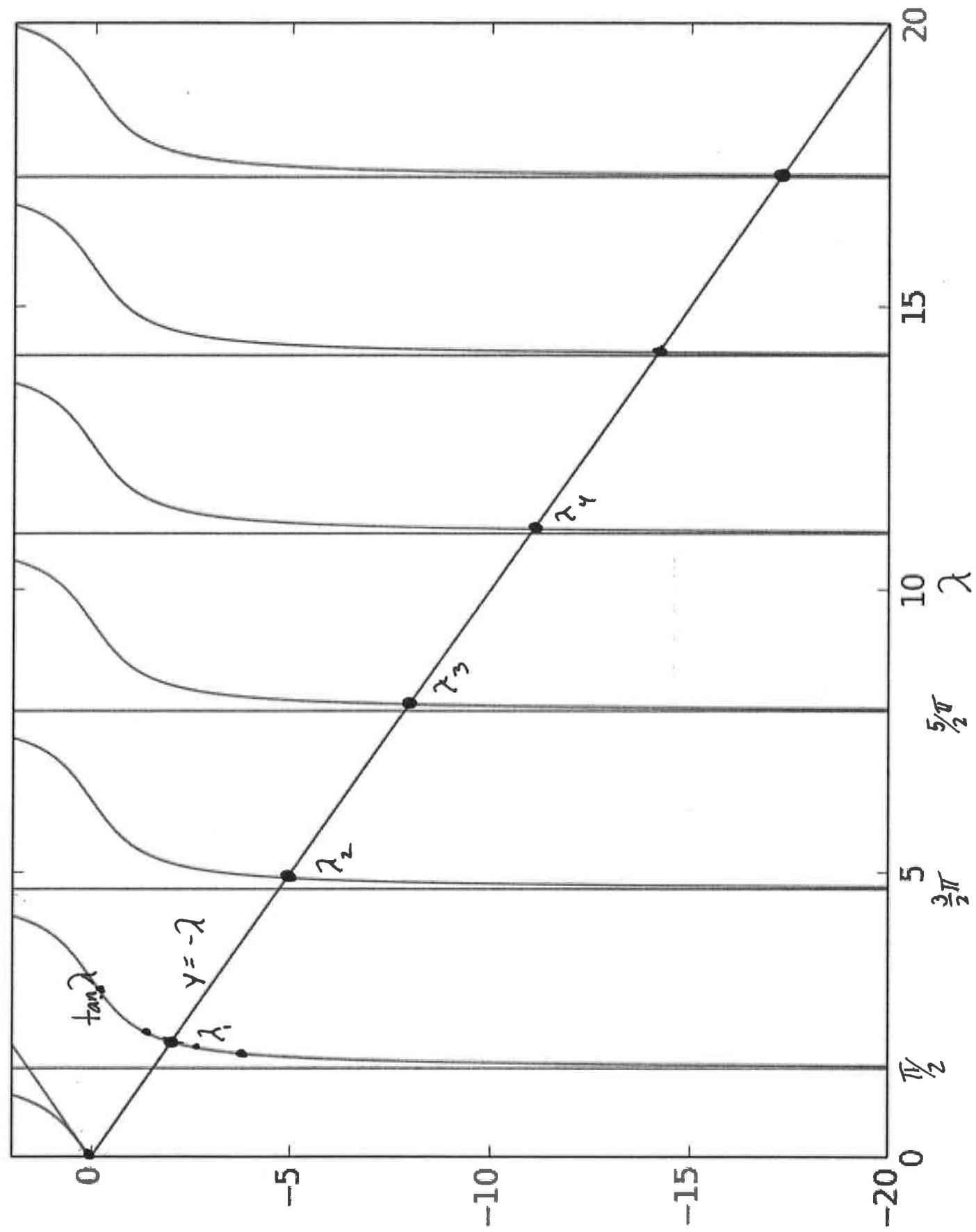
$$\boxed{\lambda \cos \lambda + h \sin \lambda = 0}$$

n	Exact λ_n	Approximate $\approx \frac{\pi}{2} + \pi(n-1)$
1	2.02	1.57
2	4.91	4.71
3	7.98	7.85
4	11.08	10.99
5	14.20	14.14

$$\boxed{\lambda + h \tan \lambda = 0}$$

See figure

$$+\tan \lambda = -\lambda$$



Solutions are of the form

$$U = X^T$$
$$U = \sum_{i=1}^N e^{-\lambda_i^2 \alpha^2} A_i \sin(\lambda_i x)$$

$$U = U_0 + U = \left(\frac{1-h}{1+h} \right) x + l + \sum_{i=1}^N e^{-\lambda_i^2 \alpha^2} A_i \sin(\lambda_i x)$$

How do we find A_i ? Same as before.

$$\phi(x) = A_i \sin(\lambda_i x)$$

Integrate with ^{premultiplied} trial function $\sin(\lambda_j x)$

$$\int_0^l \sin(\lambda_j x) \phi(x) dx = \int_0^l \sin(\lambda_j x) A_i \sin(\lambda_i x) dx$$
$$= A_i \frac{\lambda - \sin \lambda \cos \lambda}{\lambda}$$

$$A_i = \frac{\lambda_i}{\lambda_i - \sin \lambda_i \cos \lambda_i} \int_0^l \sin(\lambda_i x) \phi(x) dx$$

yes, this is a complicated solution.

Why would you do this?!?