

GES 554

Lecture 5

Sturm - Liouville

Sturm Liouville Differential Equations. (Theory ...)

$$\boxed{\begin{aligned} & (P(x)y')' - q(x)y + \lambda r(x)y = 0 & y(x) \\ & a_1y(0) + a_2y'(0) = 0 \\ & b_1y(1) + b_2y'(1) = 0 \end{aligned}}$$

Charles François Sturm (mid 1800s) and Joseph Liouville (1800s)

- Example L6P1 (from the last lecture)

$$\begin{array}{lll} U_t = \alpha^2 U_{xx} & \xrightarrow{\text{Sep Vars}} & U_t = \alpha^2 U_{xx} \\ U(0,t) = 1 & \Rightarrow & U(0,t) = 0 \\ U_x(1,t) + hU(1,t) = 1 & & U_x(1,t) + hU(1,t) = 0 \end{array}$$

Sep of Vars

$$T'_t(t) = -\lambda^2 \alpha^2 T(t) \quad \leftarrow \text{ODE}$$

$$X''_x(x) = -\lambda^2 X(x) \quad \leftarrow \text{S-L ODE}$$

$$\begin{array}{l} P(x) = 1 \\ q(x) = 0 \\ \lambda_{n_k} = +\lambda^2 \\ r(x) = 1 \end{array} \Rightarrow (X')' + \lambda_{n_k} X = 0$$

Yes, the separated spatial function X is a S-L problem

- We call the ordered set of λ eigenvalues.

The set of solutions $y(x)$ or $\phi(x)$ are called eigenfunctions.

Each eigenfunction has an eigenvalue.

N.B. "eigen" in German means "self".

$$y' = \frac{dy}{dx}$$

Lagrange's Identity

- Often, the $-(p(x)y')' + q(x)y$ part of S-L is condensed as a linear operator $L(y)$

$$\boxed{L(y) = \lambda r(x)y} \text{ is SL problem}$$

- Return to our old standby: 1) multiply by a trial function and 2) integrate

$$\int_0^1 L(u)v \, dx = \int_0^1 (-pu')'v + quv \, dx$$

Integrate the RHS by parts twice (you should do this at home)

$$\int_0^1 L(u)v \, dx = \int_0^1 u L(v) \, dx - p(x)(u'(x)v(x) - u(x)v'(x)) \Big|_0^1$$

Apply the BCs and magic!

$$\int_0^1 L(u)v \, dx = \int_0^1 u L(v) \, dx$$

- The integration and multiplication operation is called the:

Inner Product
$(u, v) = \int_0^1 u(x) \bar{v}(x) \, dx$

\bar{v} is complex conjugate
and is
 $\bar{v} = v$ when v is real only

- When

$$(L(u), v) = (u, L(v))$$

the problem is called self-adjoint

The Sturm-Liouville problem has some important properties

Theorem

All eigenvalues are real $(\lambda_n \text{ are real})$

Proof:

Let the eigenvalues be complex $\lambda = \mu + i\nu$

Let the solution be complex $\phi = U(x) + iV(x)$

Using Lagrange's Identity $(U, V) = \int_0^1 U(x) \bar{V}(x) dx$ gives

$$(L(\phi), \phi) = (\phi, L(\phi)) \quad \text{with } L(\phi) = \lambda r \phi$$

so

~~$(\lambda r \phi, \phi) = (\phi, \lambda r \phi)$~~

Expand with the identity

$$\int_0^1 \lambda r(x) \phi(x) \bar{\phi}(x) dx = \int_0^1 \phi(x) \bar{\lambda} \bar{r}(x) \bar{\phi}(x) dx$$

$r(x)$ is real $\Rightarrow \bar{r} = r$

pull out λ and $\bar{\lambda}$ (constants)

$$\lambda \int_0^1 r(x) \phi(x) \bar{\phi}(x) dx = \bar{\lambda} \int_0^1 \phi(x) \bar{r}(x) \bar{\phi}(x) dx$$

or

$$(\lambda - \bar{\lambda}) \int_0^1 r(x) \phi(x) \bar{\phi}(x) dx = 0$$

Also

$$\phi \bar{\phi} = (U(x) + iV(x))(U(x) - iV(x)) = U^2(x) + V^2(x)$$

$$(\lambda - \bar{\lambda}) \int_0^1 r(x) (U^2(x) + V^2(x)) dx = 0$$

always positive.

It turns out that $r > 0$ for ~~the~~ most problems.
Showing this is more complicated.

so

$$\lambda - \bar{\lambda} = 0$$

$\boxed{\lambda \text{ is Real}}$

Theorem

The S-L eigenfunctions are orthogonal

This means for ^{any} two eigenfunctions ϕ_1 and ϕ_2 ($\phi_1 \neq \phi_2$)

$$\int_0^l r(x) \phi_1(x) \phi_2(x) dx = 0$$

You can prove this with a similar method to proving real eigenvalues.

Theorem

The S-L eigenvalues are ordered, map 1:1 with eigenfunctions, and ~~continues to ∞~~ ^{has infinitely many}

$$\lambda_1 < \lambda_2 < \lambda_3 \dots < \lambda_n < \dots \lambda_\infty$$

$\downarrow \quad \downarrow \quad \downarrow \quad \dots \quad \downarrow \quad \dots$

$$\phi_1 \quad \phi_2 \quad \phi_3 \quad \dots \quad \phi_n \quad \dots$$

- Much like eigenvectors from linear algebra, we can scale ϕ_n .

The canonical form of ϕ_n is normalized ($\int_0^l r(x) \phi_n^2(x) dx = 1$).
 We call these an orthonormal set.

- We will pick up further properties in a later part (wave equations) of the class.

You might need a numerical routine to find λ .

From our problem (L6P1)

$$\lambda \cos \lambda + h \sin \lambda = 0 \quad \Rightarrow \quad \lambda + h \tan \lambda = 0$$

- Use Newton's method

$$\text{Residual} = R = \lambda \cos \lambda + h \sin \lambda$$

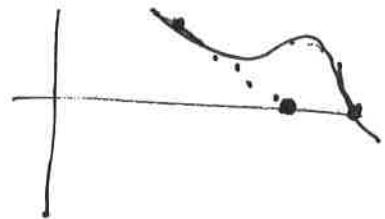
$$\frac{dR}{d\lambda} = \cos \lambda - \lambda \sin \lambda + h \cos \lambda$$

From Taylor Series

$$R(\lambda + \Delta \lambda) = R(\lambda_0) + \frac{dR}{d\lambda}(\lambda_0) \Delta \lambda$$

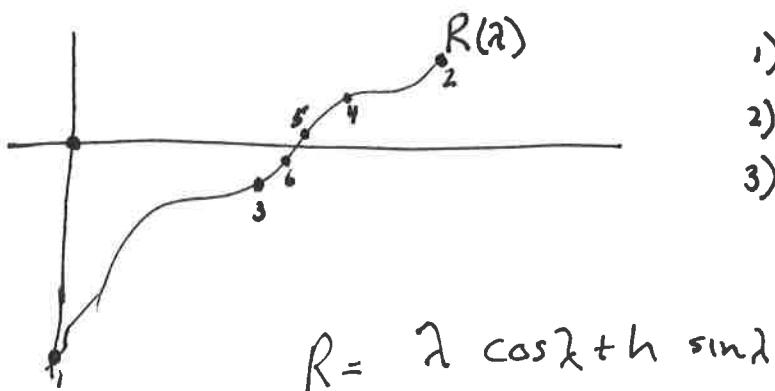
Solve for $\Delta \lambda$

$$\Delta \lambda = -\frac{R(\lambda_0)}{\frac{dR}{d\lambda}(\lambda_0)} \Rightarrow \lambda_n^{(i+1)} = \lambda_n^{(i)} + \Delta \lambda$$



Start with a good guess and iterate to find λ_m .

- Bisection



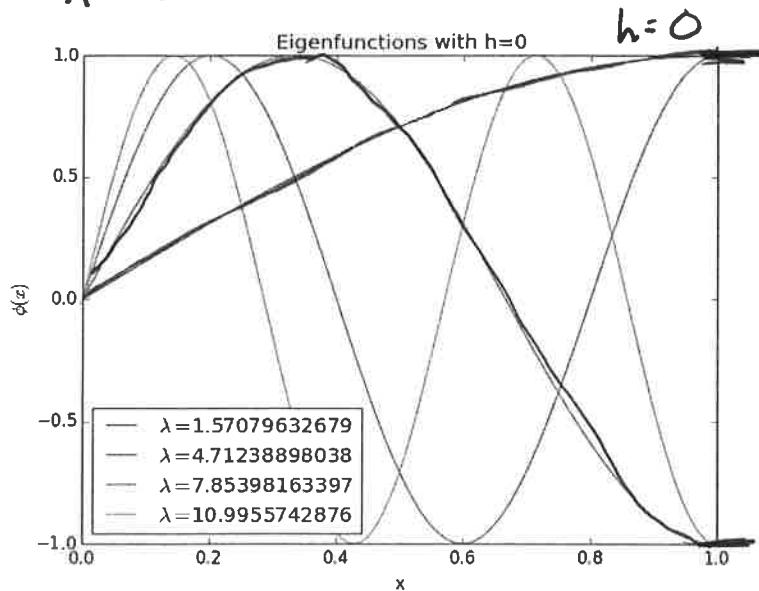
- 1) Bound the root.
- 2) Halve the domain
- 3) iterate

$$R = \lambda \cos \lambda + h \sin \lambda$$

$$\lambda + h \tan \lambda = 0$$

Boundary Condition
 $U_x(1,t) + h U(1,t) = 0$

$$U_x = 0$$



$$\sin\left(\frac{\pi}{2}x\right)$$

$$\sin(1.57x)$$

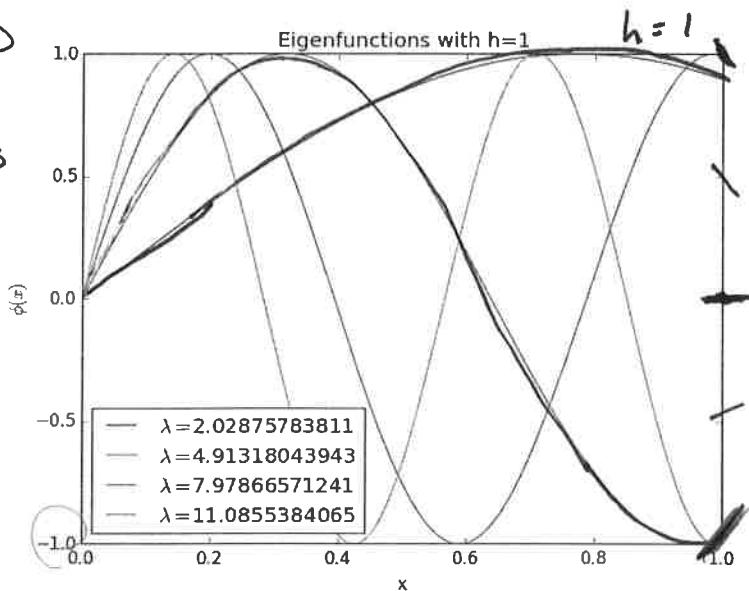
$$\sin(2x)$$

$$\lambda + h \tan \lambda = 0$$

$$U_x(1,t) + R^1 U(1,t) = 0$$

$$U_x + U = 0$$

$$U = -U_x$$



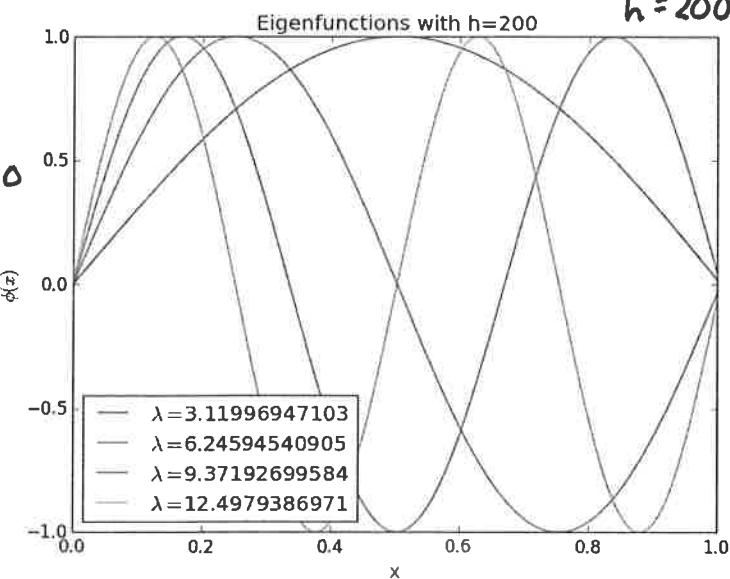
$$h=1$$

$$U_x(1,t) + 200 U(1,t) = 0$$

$$U_x + 200 U = 0$$

\approx

$$U = 0$$



How to design programs/code fast and without bugs.

- 1) What is the need? purpose?
- 2) Decompose into fundamental components
- 3) Write an operation or function to compute each component
- 4) Connect lower-order functions to generate higher order functions
- 5) Test each function as you write it (keep the tests)
- 6) Profile the code to find slow functions. Only fix the slow parts.
(Pareto's rule : 80% ~~of results~~ determined by 20% of actions)
64% of results determined by 4% of actions
- 7) Provide an example case (small). Documentation?
- 8) Did you meet the need in #1? Submit or present.
- 9) Move on to a new project.

iterate

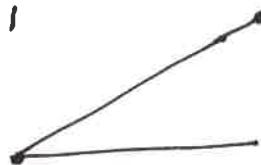
L7 P3

$$U_t = U_{xx} \quad 0 < x < 1$$

$$U_x(0, t) = 0$$

$$U_x(1, t) = 0$$

$$U(x, 0) = X$$



Sep of Vars

$$U = XT \Rightarrow X'T_t = X_{xx}T \Rightarrow \frac{T_t}{T} = \frac{X_{xx}}{X} = -\lambda^2$$

$$T(t) = e^{-\lambda^2 t}$$

$$X = A \sin(\lambda x) + B \cos(\lambda x)$$

Apply BCs

$$X_x(0) = 0 = \lambda A \cos(0) - B \lambda \sin(0) \Rightarrow A = 0$$

$$X_x(1) = 0 = 0 - B \lambda \sin(\lambda) \Rightarrow \text{either } \lambda = 0 \text{ or } \sin \lambda = 0$$

$$\text{so } \lambda = n\pi$$

$$X = B \cos(n\pi x)$$

Apply IC

$$\phi(x) = b_0 \cos(n\pi x)$$

mult by $\cos(m\pi x)$ and integrate (or be crafty and look up $F_n(x)$!)

$$\int_0^1 \cos(n\pi x) \times dx = b_0 \underbrace{\int_0^1 \cos(n\pi x) \cos(m\pi x) dx}_{\text{we know this is non-zero only when } n \neq m} = b_0 \frac{1}{2} \text{ when } n \neq 0$$

$$b_n = 2 \int_0^1 \cos(n\pi x) \times dx \text{ when } n \neq 0$$

$$b_n = \begin{cases} \frac{1}{2} & n=0 \\ \frac{2(-1)^n}{n^2\pi^2} - \frac{2}{n^2\pi^2} & n \neq 0 \end{cases}$$

$$b_0 = \frac{1}{2} \quad b_1 = -\frac{2}{\pi^2} \quad b_2 = 0$$

$$b_3 = -\frac{4}{9\pi^2}$$

Discussion

- The b_0 term of a cosine series might not be zero.
Don't forget the constant offset!
- The steady state solution can be found by inspection!!
 $u(x, \infty) = 0.5$
Why?