

GES 554

Lecture 6

Transform Hard problems

## Transform Hard Equations into Easier Ones (Lesson 8)

$$U_t = \alpha^2 U_{xx} - \beta w \quad \text{Diffusion with lateral heat loss}$$

We previously looked at the steady state solution.

$$U(x, t=\infty) = \cosh\left(\sqrt{\frac{\beta}{\alpha^2}}x\right) + \frac{1 - \cosh\left(\sqrt{\frac{\beta}{\alpha^2}}x\right)}{\sinh\sqrt{\frac{\beta}{\alpha^2}}x} \sinh\sqrt{\frac{\beta}{\alpha^2}}x$$

Some of you found a more intuitive way to write this

$$U(x, t=\infty) = \frac{e^{cx} + e^{c(1-x)}}{e^c + 1}$$

- What if  $\alpha = 0$ ?

$$\tilde{U}_t = -\beta w \Rightarrow \tilde{U}(t) = e^{-\beta t} = \dots$$

- So Decompose into

$$U = \tilde{U}w = e^{-\beta t}w$$

Subs into Gov Egu.

$$U_t = e^{-\beta t}w_t - \beta e^{-\beta t}w$$

$$U_{xx} = w_{xx} e^{-\beta t}$$

$$U = \dots$$

so

$$\underbrace{e^{-\beta t}w_t - \beta e^{-\beta t}w}_{=0} = \alpha^2 w_{xx} e^{-\beta t} - \underbrace{\beta e^{-\beta t}w}_{=0}$$

And  $e^{-\beta t}$  is never 0

$$\boxed{w_t = \alpha^2 w_{xx}} \quad \text{No lateral heat loss term now!}$$

Solve the simplified Gov Ego as before.

Total Soln

$$U = e^{-\beta t}w$$

# Important Result (Theorem)

$\sin(n\pi x)$  is orthonormal

What does this mean?

$$\text{Orthonormal} \equiv \int_0^1 \phi_1(x) \bar{\phi}_2(x) dx = 0 \quad \text{when } \phi_1 \neq \phi_2 \\ = 1 \quad \text{when } \phi_1 = \phi_2$$

What does this really mean?

You will need to solve  $\int_0^1 \sin(n\pi x) \sin(m\pi x) dx$  often.

$n$  and  $m$  are integers!

- Integration by parts is a mess.
- Tabular Int' by parts has no ~~is~~ zero stopping point
- Your calculator will trick you! Demo. Warning!
- The solution is simple.  $\frac{1}{2}$  when  $m=n$  and 0 otherwise.

Soln

$$\text{When } m \neq n, \int_0^1 \dots = \left. \frac{\sin(m\pi x)n \cos(n\pi x) - m \cos(m\pi x) \cdot \sin(n\pi x)}{m^2\pi^2 - n^2\pi^2} \right|_0^1 = 0$$

when  $m=n$

$$\int_0^1 \dots = \frac{1}{2} - \left. \frac{\sin(n\pi x) \cos(n\pi x)}{2n\pi} \right|_0^1$$

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} \frac{1}{2} & \text{when } n=m \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_n = 2 \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 1 & \text{when } n=m \\ 0 & \text{otherwise} \end{cases}$$

# Review of Integrating Factors from Diff' Eqs.

Let  $y_+ + ay = g(t)$

- Integrating factor method

1) pick a term seen in solution of homogeneous solution  $e^{at}$

2) multiply ODE by ~~integrating~~ factor/term.

$$e^{at}y_+ + e^{at}ay = e^{at}g(t)$$

3) Notice that

$$e^{at}y_+ + e^{at}ay = \frac{d}{dt}(e^{at}y)$$

4) Substitute and notice that all "y" terms are on LHS.

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

5) Integrate (with  $\tau$  up to time  $t$ )

$$\int_0^t \frac{d}{d\tau}(e^{a\tau}y) d\tau = \int_0^t e^{a\tau}g(\tau) d\tau$$

$$\underbrace{\Phi \cdot \Phi^{-1}}_1 = 1 \quad \int \frac{d}{d\tau} d\tau \text{ "cancel".}$$

$$6) e^{a\tau}y|_0^t = \int_0^t e^{a\tau}g(\tau) d\tau$$

7) Apply BC/IC and simplify

$$e^{at}y(t) - e^0 y(0) = \int_0^t e^{a\tau}g(\tau) d\tau$$

8) Final Result

$$y(t) = e^{-at} \int_0^t e^{a\tau}g(\tau) d\tau + e^{-at}y(0)$$

EE: should be  
thinking  
Convolution  
integral...  
Also  
 $\int e^{-a(t-\tau)} g(\tau) d\tau$

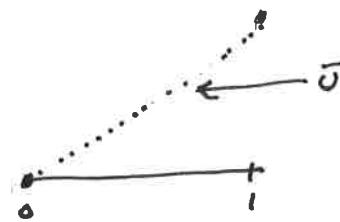
# L8 P2

$$U_+ = U_{xx} - U + X$$

$$U(0,t) = 0$$

$$U(1,t) = 1$$

$$U(x,0) = 0$$



a) Change the nonhomogeneous BCs to homogeneous ones

- Standard approach is to separate the solution

$$U = \bar{U} + U$$

Is it necessary that  $\bar{U}$  is the steady state solution? No

Is it necessary that  $\bar{U}$  fits the BCs? Yes

Pick  $\bar{U} = X$ .

Substitute

$$\bar{U}_+ + U_+ = \bar{U}_{xx} + U_{xx} - \bar{U} - U + X$$

$$U_+ = U_{xx} - U$$

BCs

$$\bar{U}(0) + U(0)t = 0$$

$$U(0,t) = 0$$

$$\bar{U}(1) + U(1,t) = 1$$

$$U(1,t) = 0$$

IC

$$\bar{U}(x,0) + U(x,0) = 0$$

$$U(x,0) = -X$$

We traded a homogeneous IC for a homogeneous BC.

• How do we get rid of the  $-U$  term?

What if we had a modified gov-eq?

$$U_t = \alpha U_{xx} - U$$

If  $\alpha = 0$  then the solution is  $U = e^t$

Let the solution be separated again (i.e. pull  $e^t$  out of  $U_{\text{solution}}$ )

$$U = F(t) \cdot w(x,t) \text{ and use } F = e^t$$

Subs' into the actual gov-eq (not the simplified one where  $\alpha = 0$ )  
o, only a function of  $t$ .

$$F_t w + F w_t = F_{xx} w + F w_{xx} - F w$$

Subs' def' of  $F$

$$-e^t w + e^t w_t = \cancel{e^t w_{xx}} - \cancel{e^t w} \Rightarrow e^t w_t = e^{-t} w_{xx}$$

We know that  $e^t$  is always nonzero for  $0 < t < \infty$ , so divide by  $e^t$

$$e^t (w_t - w_{xx}) = 0 \Rightarrow \boxed{w_t - w_{xx} = 0}$$

We can solve this!

BC.

$$F(t=0) \underset{\substack{\text{to} \\ \infty}}{} w(0,t) = 0 \quad \text{Always nonzero}$$

$$\Rightarrow \boxed{w(0,t) = 0}$$

$$F(t=\infty) \underset{\substack{\text{to} \\ 0}}{} w(1,t) = 0 \quad \text{Always nonzero}$$

$$\Rightarrow \boxed{w(1,t) = 0}$$

IC.

$$F(t=0) \overset{e^0=1}{=} w(x,0) = -x$$

$$\Rightarrow \boxed{w(x,0) = -x}$$

Solve  $\omega(x,t) = \text{Sep at Vars}$

$$\omega = TX$$

$$T_t X = TX_{xx} \Rightarrow \frac{T_t}{T} = \frac{X_{xx}}{X} = -\lambda^2$$

Solutions

$$1) T(t) = e^{-\lambda^2 t}$$

$$2) X_{xx} + \lambda^2 X = 0 \Rightarrow X(x) = A \sin(\lambda x) + B \cos(\lambda x)$$

The  $X$  equation is a form of the Sturm-Liouville problem.

$\lambda$  are real, discrete, and countable to  $\infty$

BCs are

$$X(0) = 0 \quad \text{and} \quad X(1) = 0 \Rightarrow B = 0$$

$$A \sin(\lambda) = 0$$

$$\text{Thus } \sin(\lambda) = 0 \Rightarrow \lambda = n\pi$$

~~even~~

IC.

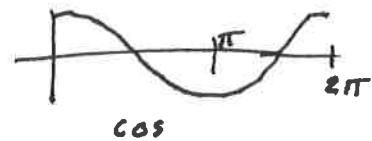
$$A_n = 2 \int_0^1 (-x) \sin(n\pi x) dx = 2 \int_0^1 (-x) \sin(n\pi x) dx$$

Integration by parts

$$\begin{array}{c|cc} & F & G \\ \hline + & -x & \sin n\pi x \\ - & -1 & -\frac{1}{n\pi} \cos n\pi x \\ + & 0 & -\frac{1}{n^2\pi^2} \sin n\pi x \end{array}$$

$$A_n = 2 \left[ \frac{x}{n\pi} \cos n\pi x - \frac{1}{n^2\pi^2} \sin n\pi x \right] \Big|_0^1$$

$$= \frac{2}{n\pi} \cos n\pi - \frac{2}{n^2\pi^2} \sin n\pi$$

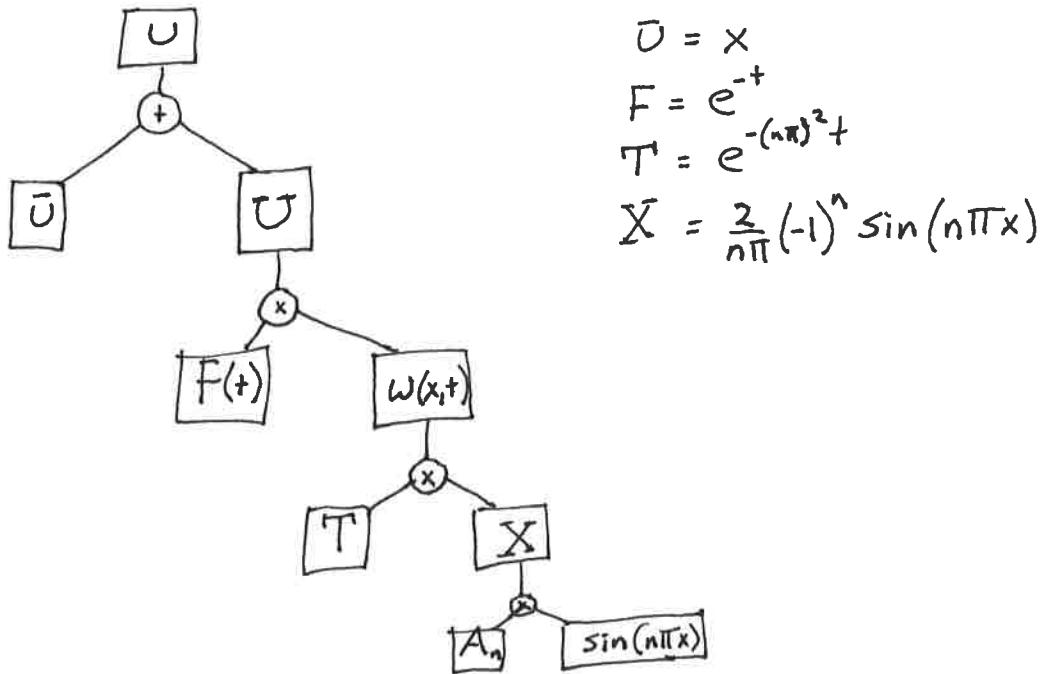


$$A_{n \text{ odd}} = -\frac{2}{n\pi}$$

$$A_{n \text{ even}} = \frac{2}{n\pi}$$

$$A_n = \frac{2}{n\pi} (-1)^n$$

Now, fit the decomposition back together



$$U = \bar{U} + FTX$$

$$\begin{aligned}\bar{U} &= x \\ F &= e^{-t} \\ T &= e^{-(n\pi)^2 t} \\ X &= \frac{2}{n\pi} (-1)^n \sin(n\pi x)\end{aligned}$$

$$U = x + e^{-t} e^{-\pi^2 t} \frac{2}{n\pi} (-1)^n \sin(\pi x) + e^{-t} e^{-4\pi^2 t} \frac{1}{\pi} \sin(2\pi x) + \dots$$

contours of  $u$

