

GES 554

Lecture 7

Non homogeneous PDEs

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Integral Transforms (intro)

1st Exam

5th Feb 2016

$$1 \text{ hr} = 60 \text{ min}$$

Covers up to lesson 10 in Farlow

Fourier Toy

Sawtooth

Non Homogeneous PDE

$$\underbrace{L(u)}_{\sim} = f(x, t)$$

$u_t - \alpha^2 u_{xx}$ if diffusion

Solution through Eigenfunction Expansion

1) Find eigenfunctions of the homogeneous PDE

$$\lambda_n, X_n(x)$$

2) Expand PDE as

$$u = T_n(t) X_n(x)$$

this will require writing $f(x)$ as $X(x)$

3) Solve resulting T_n differential equation

- Substitute $u = T_n(t) X_n(x)$ into PDE
- Separate out the $X(x)$ terms
- Find T_n ODE

4) Fit back together

$$u = T_n(t) X_n(x)$$

Lesson 9 Problem 5 "Time varying"

$$U_t = U_{xx}$$

$$U(0,t) = 0$$

$$U(1,t) = \cos t$$

$$U(x,0) = X$$

Gov Eqs are homogeneous

BCs are non-homogeneous

- Decompose via $U = \bar{U} + U$ with something that fits BCs $\bar{U} = \cos(t)x$

$$\bar{U}_t + U_t = \bar{U}_{xx} + U_{xx}$$

$$-\sin(t)x + U_t = 0 + U_{xx} \Rightarrow$$

$$U_t = U_{xx} + \sin(t)x$$

$$\text{BCs } \bar{U}(0,t) + U(0,t) = 0$$

$$U(0,t) = 0$$

$$\bar{U}(1,t) + U(1,t) = \cos t$$

$$U(1,t) = 0$$

IC

$$\cancel{\bar{U}(x,0)} + U(x,0) = x \Rightarrow$$

$$U(x,0) = 0$$

Now the gov eq is non-homogeneous but still a PDE

- Separation of Vars' for the homogeneous case to find eigenfunctions

$$U = XT \Rightarrow X_{xx}T' = XT \Rightarrow \frac{X_{xx}}{X} = \frac{T'}{T} = -\lambda^2$$

We really only care about the form of X

$$X = A \sin(\lambda x) + B \cos(\lambda x)$$

BCs are

$$X(0) = 0 \Rightarrow B = 0$$

$$X(1) = 0 \Rightarrow A \sin(\lambda x) = 0$$

This is a S-L problem. If $A \sin(\lambda) = 0$ and $A \neq 0$, then $\lambda = n\pi$

The eigenfunctions are $X_n = \sin(n\pi x)$ with $\lambda_n = n\pi$

- Notice that the IC is 0. We can shortcut the IC response.

There is no transient in \bar{U} based on ICs

• Subs $U = T_n X_n$ into BCs

$$U(0,+) = 0 = T_n(+) \cancel{X_n(0)}^0 \Rightarrow \text{Useless}$$

$$U(1,+) = 0 = T_n(+) \cancel{X_n(1)}^0 \Rightarrow \text{Useless}$$

$$U(x,0) = 0 = T_n(0) \cancel{X_n(x)}^{\text{nonzero}} \Rightarrow T_n(0) = 0$$

- Solutions of \bar{U} will contain the eigenfunction

$$\bar{U} = T_n X_n \quad (\text{implied summation of } n=1, \infty)$$

Substitute into the Gov Egu? Not yet

- Forcing Function ~~is~~ represented in terms of eigenfunctions.

$$f(x) = \sin(t) x = F_n X_n$$

Multiply by X_m and integrate

$$\int_0^1 f(x) \sin(m\pi x) dx = \int_0^1 F_n \sin(n\pi x) \sin(m\pi x) dx \xrightarrow{\frac{1}{2} \text{ when } n=m}$$

$$F_n = 2 \int_0^1 \underbrace{\sin(t) x \sin(n\pi x)}_G dx$$

don't forget about this term!

$$\begin{array}{ccc} + & F & G \\ + & x & \sin(n\pi x) \\ - & 1 & \frac{-1}{n\pi} \cos(n\pi x) \\ + & 0 & \frac{-1}{n^2\pi^2} \sin(n\pi x) \end{array}$$

$$f_n = -\frac{2}{n\pi} \cos(n\pi x) + 0 = -\frac{2}{n\pi} (-1)^n$$

$$f(x) = -\frac{2}{n\pi} (-1)^n \sin(n\pi x) \sin(t) = f_n \sin(n\pi x) \sin(t)$$

- Substitute $\bar{U} = T_n X_n$ and $f(x)$ into Gov Egu.

$$T_{n+} X_n = T_n X_{n+} + f_n \sin(n\pi x) \sin(t)$$

This needs X_n substituted... where $X_n = \sin(n\pi x)$

$$T_{n+} \sin(n\pi x) = -T_n n^2 \pi^2 \sin(n\pi x) + f_n \sin(t) \sin(n\pi x)$$

Collect $\sin(n\pi x)$ terms and recognize that $\sin(n\pi x)$ is nonzero on $0 < x < 1$

$$\sin(n\pi x) [T_{n+} + T_n n^2 \pi^2] = f_n \sin(t)$$

The ODE with a forcing function for T_n is

$$T_{n+} + n^2 \pi^2 T_n = f_n \sin(t)$$

- Solution of T_n

An integrating factor method seems useful. Multiply by $e^{ct} = e^{n^2\pi^2 t}$

$$e^{n^2\pi^2 t} T_n + n^2\pi^2 e^{n^2\pi^2 t} T_n = e^{n^2\pi^2 t} f_n \sin(t)$$

$$\frac{d}{dt}(e^{n^2\pi^2 t} T_n) = \dots$$

Integrate (change t to s inside integral to avoid confusing t with \dot{t} .)

$$\int_0^t \frac{d}{ds}(e^{n^2\pi^2 s} T_n) ds = \int_0^t e^{n^2\pi^2 s} f_n \sin(s) ds$$

Divide by $e^{n^2\pi^2 t}$

$$e^{n^2\pi^2 t} T_n(t) - e^0 T_n(0) = f_n \int_0^t e^{n^2\pi^2 s} \sin(s) ds$$

$$T_n(t) = e^{-n^2\pi^2 t} T_n(0) + e^{-n^2\pi^2 t} f_n \int_0^t e^{n^2\pi^2 s} \sin(s) ds$$

- Total Solution

$$U = \bar{U} + \bar{U}' = \bar{U} + T_n X_n$$

$$U = \cos(t) X + e^{-n^2\pi^2 t} \left(\frac{-2}{n\pi} \right) (-1)^n \left(\int_0^t e^{n^2\pi^2 s} \sin(s) ds \right) \sin(n\pi X)$$

Integral Transforms.

$$F(s) = \int_A^B K(s, t) f(t) dt$$

↑
Kernel

A transform pair also exists (for a particular transform)

$$\mathcal{F}^{-1}(F) = f(t)$$

See Farlow page 73 , Tabulated in Appendix 1

When applied to PDEs, the transform makes the solution easier to obtain.

L10 p1. Prove 10.1 What assumption is required for f ?

$$1) \quad \mathcal{F}'_s(f') = -\omega \mathcal{F}'_c(f)$$

$$\text{Definition of } \mathcal{F}_s = \frac{2}{\pi} \int_0^\infty f(t) \sin(\omega t) dt$$

Apply f'

$$\mathcal{F}_s(f') = \frac{2}{\pi} \int_0^\infty f'(t) \underbrace{\sin(\omega t)}_{U} dt$$

Int by Parts

$$\begin{aligned} & \text{Integration by Parts} \\ & \int u v' dx = u v - \int u' v dx \\ & \begin{aligned} v' &= f' \\ U &= \sin(\omega t) \end{aligned} \Rightarrow \begin{aligned} v &= f \\ U' &= \omega \cos(\omega t) \end{aligned} \end{aligned}$$

$$\mathcal{F}_s(f') = \frac{2}{\pi} \left(f \sin(\omega t) \Big|_0^\infty - \underbrace{\int_0^\infty f \omega \cos(\omega t) dt}_{\mathcal{F}'_c(f) \cdot \omega} \right)$$

Apply ~~the~~ integration limits

$$\mathcal{F}_s(f') = \frac{2}{\pi} \left(f(\infty) \sin(\infty) - f(0) \sin(0) \right) - \omega \mathcal{F}'_c(f)$$

this must be zero, implies $f(\infty) = 0$

$$\boxed{\mathcal{F}'_s(f') = -\omega \mathcal{F}'_c(f)}$$

The function decays to zero at large times.

$$2) \quad \mathcal{F}'_s(f'') = \frac{2}{\pi} \omega f(0) - \omega^2 \mathcal{F}'_s(f)$$

Proceed as above except Int by Parts is applied twice

$$\mathcal{F}'_s(f'') = \frac{2}{\pi} \int_0^\infty f'' \sin(\omega t) dt = \frac{2}{\pi} \left(f' \sin(\omega t) \right) \Big|_0^\infty - \frac{2}{\pi} \int_0^\infty f' \omega \cos(\omega t) dt$$

Apply again

$$\begin{aligned} &= \frac{2}{\pi} \left(f'(\infty) \sin(\infty) - f'(0) \sin(0) \right) - \frac{2}{\pi} \left(f(\infty) \cos(\infty) - f(0) \cos(0) \right) \\ &\quad - \omega^2 \frac{2}{\pi} \int_0^\infty f'(t) \sin(\omega t) dt \end{aligned}$$

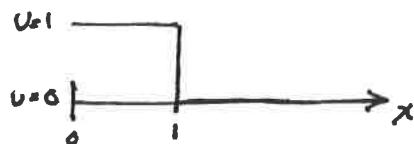
When assuming $f(\infty) = f'(0) = 0$

$$\boxed{\mathcal{F}'_s(f'') = \frac{2}{\pi} f(0) - \omega^2 \mathcal{F}'_s(f)}$$

3,4) Proceed in a similar manner.

L10P3

$$U_+ = \alpha^2 U_{xx}$$



$$U_x(0, t) = 0$$

$$U(x, 0) = H(1-x)$$

Apply Fourier cos transform (BC looks like cos more than sinc)

$$\tilde{F}_c(U_+) = \tilde{F}_c(\alpha^2 U_{xx})$$

Define $U = \tilde{F}_c(u)$

$$\frac{dU}{dt} = \alpha^2 \left[-\frac{2}{\pi} \frac{U'(0)}{\omega} - \omega^2 F_c(u) \right] = -\alpha^2 \omega^2 F_c(u) = -\alpha^2 \omega^2 U$$

Solve for $U(t)$

$$U(t) = A e^{-\alpha^2 \omega^2 t}$$

Apply $\overset{IC}{\cancel{F}_c}$ as $\tilde{F}_c(BC)$

$$\tilde{F}_c(U(x, 0)) = \tilde{F}_c(H(1-x)) \quad \text{Appendix I, Table C, Eq 6. } \alpha=1$$

$$U(0) = \frac{2}{\pi \omega} \sin(\omega)$$

Subs. for $A = U(0)$

$$U(t) = \frac{2}{\pi \omega} \sin(\omega) e^{-\alpha^2 \omega^2 t}$$

Apply inverse \tilde{F}_c^{-1} to obtain solution. (\tilde{F}^{-1} wrt x)

$$U(t) = \tilde{F}_c^{-1}(U(t)) = \int_0^\infty U(t) \cos(\omega x) d\omega = \frac{2}{\pi} \int_0^\infty \frac{\sin(\omega)}{\omega} e^{-\alpha^2 \omega^2 t} \cos(\omega x) d\omega$$

$$U(t) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\omega)}{\omega} e^{-\alpha^2 \omega^2 t} \cos(\omega x) d\omega$$

Demo L10P3.py