

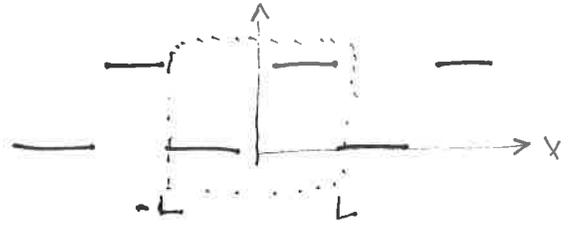
GES 554

Lecture 8

Fourier and
Laplace Transforms
+
Duhamel

L11P1

Fourier expansion of a square wave



The function is represented by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \underline{b_n \sin(n\pi x)}$$

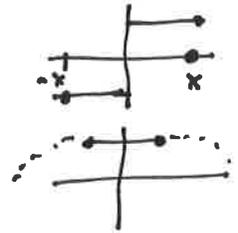
Terms are calculated from

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

We could plug + chug or we can be crafty.

- The function $f(x)$ is odd. $f(-x) = -f(x)$
- Cosine is an even function. $f(-x) = f(x)$
- An odd function times an even function is an odd function.



Cosine terms.

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \text{By Inspection!}$$

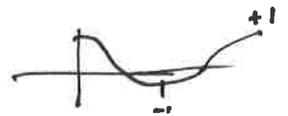
Symmetric integration of odd function = 0

Sine terms

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{This is odd} \cdot \text{odd} = \text{even}$$

Since the kernel is even, $\int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ Thus

$$b_n = 2 \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{n\pi} \cos \overset{-1 \text{ odd}}{+1 \text{ even}} \frac{n\pi x}{L} + \frac{2}{n\pi} \cos \theta$$



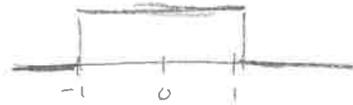
$$b_n = \frac{2}{n\pi} (1 - (-1)^n)$$

Demo

L11P4

 $F(\xi)$ What is the Fourier transform of $f(x)$?

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$



$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} e^{-i\xi x} \left(\frac{1}{-i\xi} \right) \Big|_{-1}^1$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{+i\xi} \right) \left[-e^{-i\xi} + e^{i\xi} \right] \frac{2}{2}$$

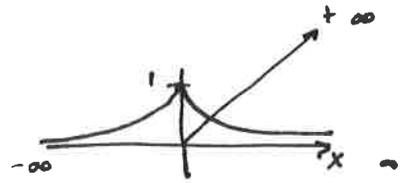
Remember that $\frac{e^{i\xi} - e^{-i\xi}}{2i} = \sin \xi$

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{\xi} \sin \xi = \boxed{\frac{2}{\sqrt{2\pi}} \operatorname{sinc} \xi = F(\xi)}$$

L12 p3

$$U_t = \alpha^2 U_{xx} \quad -\infty < x < \infty$$

$$U(x,0) = e^{-x^2} \quad -\infty < x < \infty$$



Apply \mathcal{F} to Gov Egu

$$\mathcal{F}(U_t) = \mathcal{F}(\alpha^2 U_{xx}) \Rightarrow \frac{d}{dt} \mathcal{F}(U) = -\alpha^2 \xi^2 \mathcal{F}(U)$$

Solve for $\mathcal{F}(U) = U$

$$\mathcal{F} U_t = -\alpha^2 \xi^2 U \Rightarrow U(t) = e^{-\alpha^2 \xi^2 t} U(0)$$

Apply \mathcal{F} to IC. Appendix I, Table A, Eq 6.

$$U(0) = \mathcal{F}(U(x,0)) = \mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

Subs.

$$U(t) = \frac{e^{-\omega^2/4}}{\sqrt{2}} e^{-\alpha^2 \xi^2 t} \quad \text{Actually } \omega \text{ and } \xi \text{ are equal.}$$

$$= \frac{e^{-\omega^2/4 - \alpha^2 \omega^2 t}}{\sqrt{2}} = \frac{e^{-(\frac{1}{4} + \alpha^2 t)\omega^2}}{\sqrt{2}}$$

Find the inverse Fourier transform of this.

$$t \frac{1}{4a^2} = \left(\frac{1}{4} + \alpha^2 t\right) \Rightarrow a^2 = \frac{1}{1+4\alpha^2 t} \Rightarrow a = \sqrt{\frac{1}{1+4\alpha^2 t}}$$

$$U(t) = \frac{e^{-a^2 x^2}}{\sqrt{1+4\alpha^2 t}}$$

$$a \frac{1}{a\sqrt{2}} = \frac{a}{a} \frac{1}{\sqrt{2}}$$

$$U(t) = \frac{1}{\sqrt{1+4\alpha^2 t}} e^{-x^2/(1+4\alpha^2 t)}$$

Laplace transform

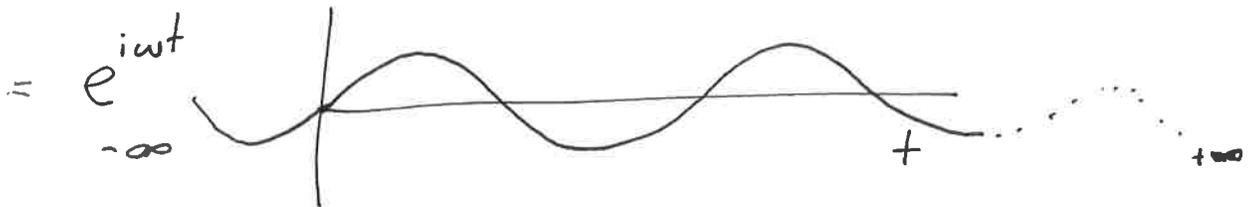
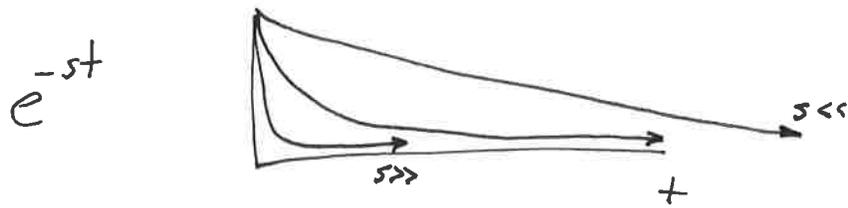
- Transforms an ODE \rightarrow algebra
- Transforms a PDE \rightarrow ODE

$$L(u) = L(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\frac{1}{\sqrt{2\pi}} \quad \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

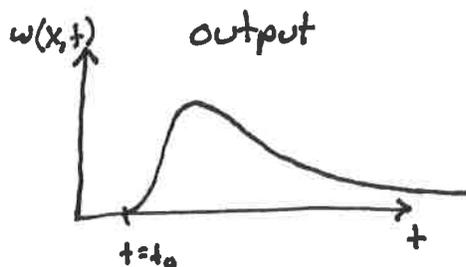
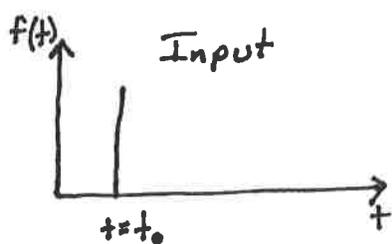
$$\frac{1}{1} \quad \frac{1}{2\pi} = \frac{1}{2\pi}$$



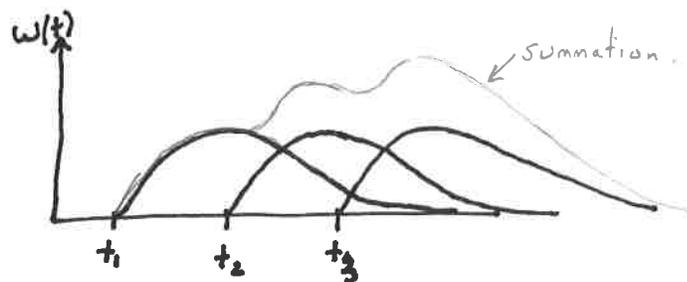
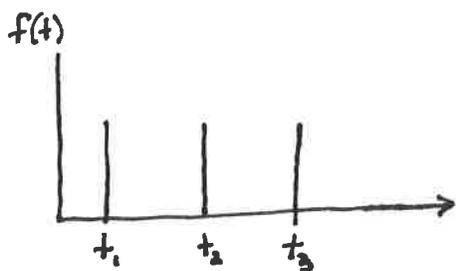
Duhamel's Principle

$$\begin{aligned}
 u(x,t) &= \int_0^t w_x(x, t-\tau) f(\tau) d\tau \\
 &= \int_0^t w(x, t-\tau) f'(\tau) d\tau + f(0) w(x,t)
 \end{aligned}$$

- Given an impulsive input and the response, we can find the response for any input.



What if we add ~~two~~ ^{three} spaced impulses?



Can we describe a continuous function $f(t)$ in terms of impulses? Yes!

$$\bullet \int_a^b f(\tau) \delta(t-\tau) d\tau = \int_a^b f(\tau) d\tau$$