

GES 554
Lecture 9

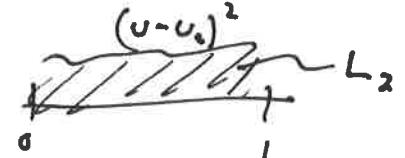
Convection terms
in
Diffusion

Interesting L_2 error term for Fourier series.

- Given a function $f(x)$, how close is our approximation?

One good measure is the L_2 error ("el squared") or Root Mean Square

$$L_2 = \int_{\Omega} (u - u_{\text{actual}})^2 dx$$



- For the Fourier series,

$$L_2 = \int_{\Omega} (a_n \sin(n\pi x) - f(x))^2 dx$$

Expand the square.

$$\begin{aligned} L_2 &= \int_{\Omega} a_n^2 \sin^2(n\pi x) dx^{0.5} \\ &\quad + \int_{\Omega} -2a_n f(x) \sin(n\pi x) dx \xrightarrow{\text{Definition of } a_n !!!} \\ &\quad + \int_{\Omega} f^2(x) dx \end{aligned}$$

$$L_2 = \frac{1}{2} a_n^2 - a_n^2 + \int_{\Omega} f^2(x) dx$$

$$L_2 = -\frac{1}{2} a_n^2 + \int_{\Omega} f^2(x) dx$$

- So for zero error

$$\sum a_n^2 = 2 \int_{\Omega} f^2(x) dx$$

Very interesting indeed!

Picard Iteration. (Something a bit different)

Given $y_x = F(x)$ and $y(0) = a$ $y = f(x)$

Integrate

$$\int_0^x y_x dx = \int_0^x F(y(x)) dx$$

$$y(x) - y(0) \int_0^x F(x) dx$$

The interesting part is that this can be iterated for $y(x)$!

$$y_{i+1}(x) = a + \int_0^x F(y_i(x)) dx$$

y_i converges to the true $y(x)$

Example:

$$y(0) = 1 \quad \frac{dy}{dx} = y \quad \text{we know the answer}$$

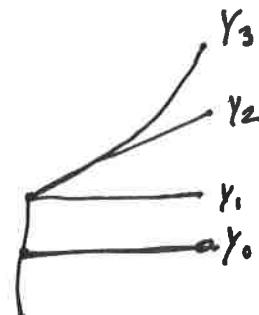
Pick $y_0 = 0$ (constant)

$$y_1 = 1 + \int_0^x 0 dx = 1$$

$$y_2 = 1 + \int_0^x 1 dx = 1 + x$$

$$y_3 = 1 + \int_0^x (1+x) dx = 1 + x + \frac{x^2}{2}$$

$$y_4 = 1 + \int_0^x (1+x) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = \lesssim \frac{x^n}{n!}$$



$y_{N \rightarrow \infty} = e^x$

Convection in the diffusion problem.

$$\underline{U_t = \underbrace{D U_{xx}}_{\text{old}} - V U_x}$$

Remember our previous discussion of Flux (how much "stuff" per second across a window)

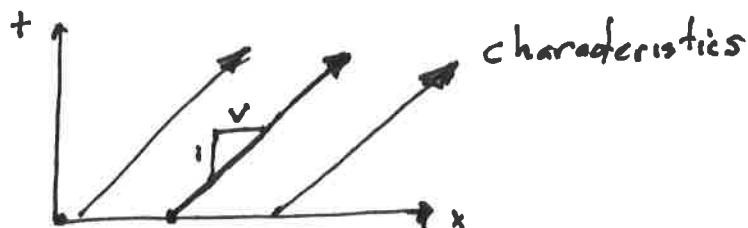


Flux in this convection problem

$$+V U_x = V \frac{du}{dx} = \frac{d}{dx}(V u)$$

or
$$\boxed{Flux = Vu}$$

The value "u" is transported at the velocity V.



So we expect a solution to be right running

$$u(x,t) = f(t - x/v)$$



Convection term U_x (Lesson 15) Review

$$U_t = \underbrace{DU_{xx}}_{\text{Diffusion}} - \underbrace{VU_x}_{\text{Convection}}$$

Convection is the transport of u due to the movement of a substance.

Flux due to convection

$$F = \overleftarrow{VU} \xrightarrow{\substack{\text{velocity of air} \\ \text{state}}} \Rightarrow U_t = -VU_x$$

Flux due to diffusion

$$F = -D \overleftarrow{U_x} \Rightarrow U_t = DU_{xx}$$

Remember that our transport equation is

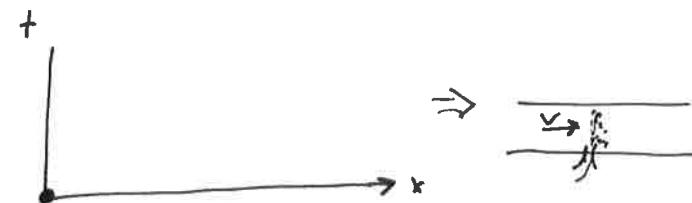
$$\boxed{\frac{du}{dt} + \frac{dF}{dx} = 0}$$

Solution of a convection problem. (Farkow p 113-114)

$$U_+ = -V U_x \quad 0 < x < \infty \quad 0 < t < \infty$$

$$U(0, t) = P$$

$$U(x, 0) = 0$$



When solving convection problems, we often look at the x-t plane.

- Hint of a solution by characteristics.

$$\frac{du}{dt} = -V \frac{du}{dx} = \frac{d(-Vu)}{dx}$$

rearrange to (chain rule)

$$\frac{du}{dt} = \frac{d(-Vu)}{du} \frac{du}{dx} \Rightarrow \frac{du}{dt} \frac{dx}{du} = \frac{d(-Vu)}{du}$$

Apparently there is a line in which $\frac{dx}{dt} = \frac{dF}{du}$

- Solution by Laplace - Transform.

$$\mathcal{L}(U_+) = \mathcal{L}(-V U_x)$$

$$\mathcal{L}(U(0, t)) = \mathcal{L}(P) = \frac{P}{s}$$

$$sU - U(0) = -V \frac{dU}{dx}$$

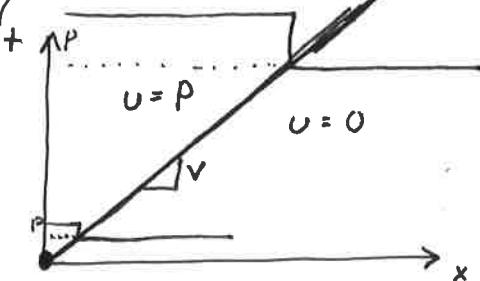
solution to U is

$$U(x) = \frac{P}{s} e^{-\frac{sx}{V}}$$

Inverse L-T is

$$U = \mathcal{L}^{-1}(U(x)) = \mathcal{L}^{-1}\left(\frac{P}{s} e^{-\frac{sx}{V}}\right) = P H(t - \frac{x}{V})$$

Visually



$$t - \frac{x}{V} = c$$

$$t - c = \frac{x}{V}$$

$$x = V(t - c)$$

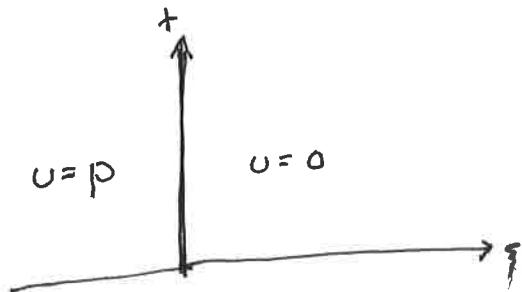
Change of Variables (Go with the flow)

$$\xi = x - vt \quad \text{New location tracking coordinate} \quad (x = \xi + vt)$$

Substitute.

$$U = \rho H\left(t - \frac{x}{v}\right) = \rho H\left(t - \frac{\xi + vt}{v}\right) = \rho H(\xi)$$

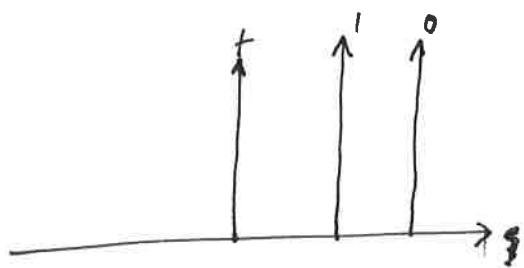
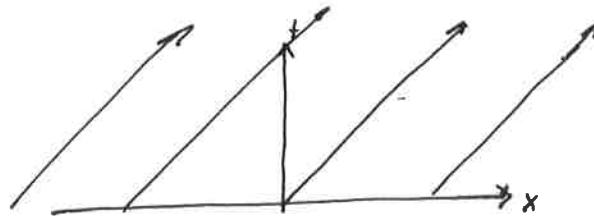
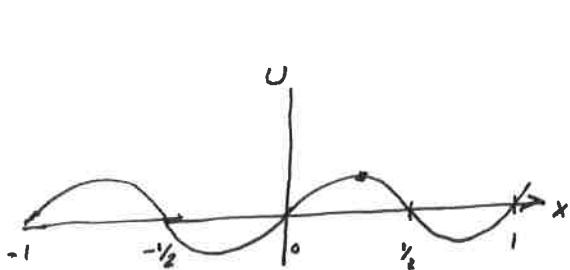
This is a Lagrangian Coordinate System.



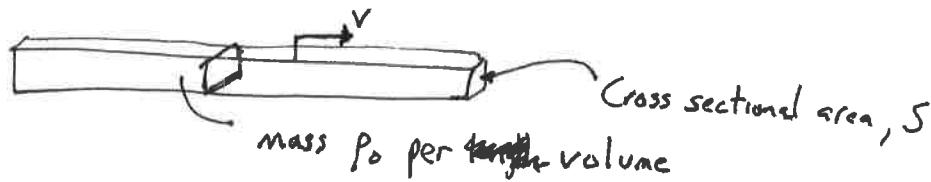
Could you solve this?

$$U_t = -vU_x$$

$$U(x, t) = \sin(x) 2\pi t$$



Longitudinal Waves in a rod



- Δx element Forces



- Newton's law

$$\sum F = ma = S \rho_0 \Delta x \frac{d^2 v}{dt^2}$$

- Stress-strain relationship

$$\sigma = \frac{F}{A} = E \frac{dv}{dx} \Rightarrow F = SE \frac{dv}{dx}$$

- Combined

$$S \rho_0 \Delta x \frac{d^2 v}{dt^2} = -F + F + \frac{dF}{dx} \Delta x \\ = \frac{d}{dx} \left(S E \frac{dv}{dx} \right) \Delta x$$

- Cancel Δx and S (if cross section is constant), divide by ρ_0

$$\boxed{\frac{d^2 v}{dt^2} = \frac{E}{\rho_0} \frac{d^2 v}{dx^2}}$$

A wave equation

$$\boxed{U_{tt} = C U_{xx}}$$

- Wave velocity is $\sqrt{\frac{E}{\rho_0}}$

- Steel bar.

$$E \approx 30 \times 10^6 \text{ psi} ; \quad \rho_0 \approx 0.28 \frac{\text{lb}}{\text{in}^3} \approx 7 \times 10^{-4} \frac{\text{slinch}}{\text{in}^3}$$

$$C \approx \sqrt{\frac{30 \times 10^6 \text{ psi}}{0.28 \text{ lb/in}^3}} \approx 17000 \frac{\text{ft}}{\text{s}}$$

$$\approx 5000 \frac{\text{m}}{\text{s}}$$

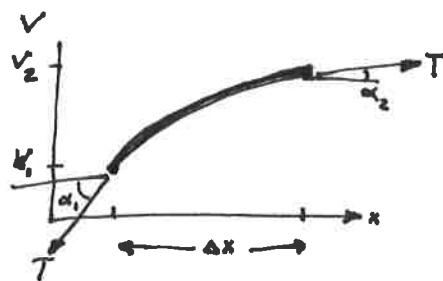
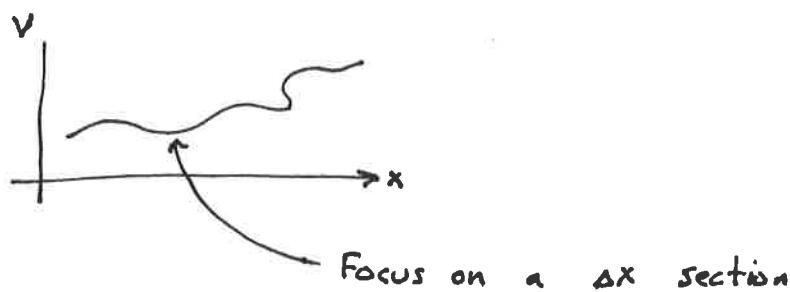
FAST

We can measure E with this!

Taut String

- Mass of λ_0 per length

- Tension T



- Newton's law in V direction on Δx section.

$$F = ma = \lambda_0 \Delta x \ddot{V}$$

- Forces in V direction on Δx section

$$F = -T \sin \alpha_1 + T \sin \alpha_2$$

But α_1 and α_2 are small, so $\sin \alpha_1 \approx \alpha_1 \approx \frac{dV}{dx}$

$$F = -T \frac{dV_1}{dx} + T \frac{dV_2}{dx}$$

- Combined

$$\lambda_0 \Delta x \ddot{V} = T \left(\frac{dV_2}{dx} - \frac{dV_1}{dx} \right)$$

- Divide by Δx and notice that $\left(\frac{dV_2}{dx} - \frac{dV_1}{dx} \right) / \Delta x$ is a finite difference.

$$\lambda_0 \ddot{V} = T \frac{d}{dx} \left(\frac{dV}{dx} \right)$$

- Canonical form

$$\ddot{V} = \frac{T}{\lambda_0} \frac{d^2V}{dx^2}$$

This is a wave equation