

GES 554

Lecture 10

Wave Equation

# Wave Equation

$$U_{tt} = \alpha^2 U_{xx}$$

$\uparrow$                      $\uparrow$   
2nd order              2nd order

$$\sqrt{\alpha} = c$$

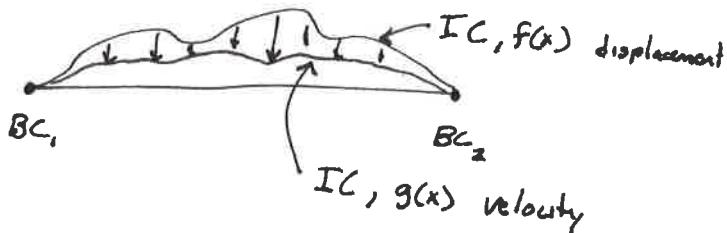
- The wave equation requires 2 initial conditions for time term.

$$U(x, 0) = f(x) \leftarrow \text{initial displacement}$$

$$U_t(x, 0) = g(x) \leftarrow \text{initial velocity}$$

- What about ~~the~~ Boundary conditions?

2 BCs.



## Separation of Variables

$$U_{tt} = \alpha^2 U_{xx} \quad a < x < b \quad 0 < t < \infty$$

$$U = X T$$

$$X T_{tt} = \alpha^2 X_{xx} T = \cancel{X T}$$

$$\frac{T_{tt}}{\alpha^2 T} = \frac{X_{xx}}{X} = -\lambda^2$$

$$T_{tt} + \lambda^2 \alpha^2 T = 0$$

$$X_{xx} + \lambda^2 X = 0 \implies X = A \sin \lambda x + B \cos \lambda x$$

$$T' = C \sin \lambda a t + D \cos \lambda a t$$



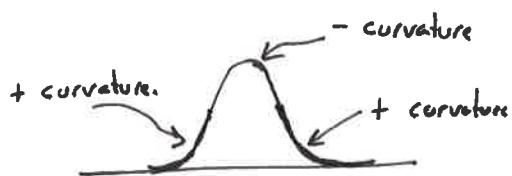
Q) How does the wave know what to do?

A) It doesn't. Waves are inanimate and don't know anything.

Q) Why does a wave behave like this?

$$U_{tt} = c^2 U_{xx}$$

the acceleration depends on the curvature



Q) How can we generate this IC?



The wave equation conserves energy. No dissipation.  $t \rightarrow \infty$

No dispersion

# L16 P4

$$U_{tt} = U_{xx}$$

- Solutions of the form

$$U(x,t) = e^{ax+bt}$$

- Substitute

$$b^2 e^{ax+bt} = a^2 e^{ax+bt}$$

- Thus  $b^2 = a^2 \Rightarrow b = \pm \sqrt{a^2} = \pm a$

- Solutions are

$$U(x,t) = A_1 e^{ax-iat} + A_2 e^{ax+iat}$$

- What if  $a^2$  is negative?

$$b = \sqrt{a^2} = i\sqrt{|a|^2} = \pm ia$$

$$a = \pm ib$$

$$U(x,t) = A_1 e^{ax+iat} + A_2 e^{ax-iat}$$

## Characteristics {Lines, Curves}

The D'Alembert solution is

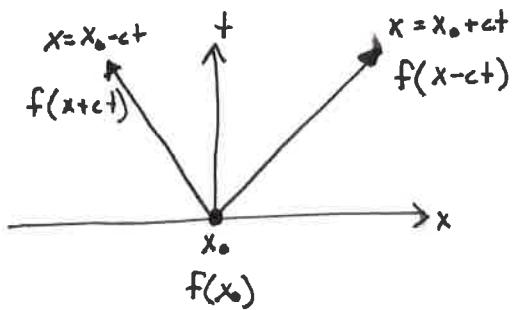
$$U(x,t) = \frac{1}{2} (f(x+ct) + f(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

Notice that the function  $f$  has an input  $x+ct$  and  $x-ct$ .

Any function moves with  $x+ct$  and  $x-ct$  for a constant value of  $f$ .

$$x_0 = x + ct \quad \text{and} \quad x_0 = x - ct$$

plot these curves



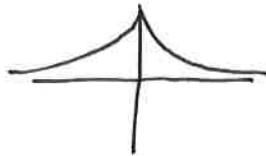
Along a characteristic, the solution <sup>function</sup> value is unchanged

L17 p3

$$U_{tt} = U_{xx}$$

$$U(x, 0) = e^{-x^2}$$

$$U_t(x, 0) = 0$$



Solution:

Apply D'Alembert

$$U(x, t) = \frac{1}{2} \left( e^{-(x - ct)^2} + e^{-(x + ct)^2} \right) + \frac{1}{2c} \int_0^t \dots$$

So simple. This is the ~~rare~~ PDE, that is so easy.

$$\begin{aligned} U(x, t) &= \frac{1}{2} e^{-(x^2 - 2xct + c^2t^2)} + \frac{1}{2} e^{-(x^2 + 2xct + c^2t^2)} \\ &= \frac{1}{2} e^{-x^2} e^{2xct} e^{-c^2t^2} + \frac{1}{2} e^{-x^2} e^{-2xct} e^{-c^2t^2} \\ &= \frac{1}{2} e^{-x^2} e^{-c^2t^2} (e^{2xct} + e^{-2xct}) \\ &= \frac{1}{2} (e^{2xct} + e^{-2xct}) e^{-(x^2 + c^2t^2)} \end{aligned}$$

Demo.

# D'Alembert Solution to W.E

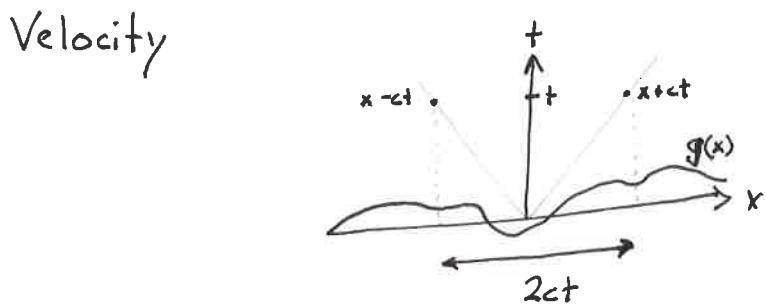
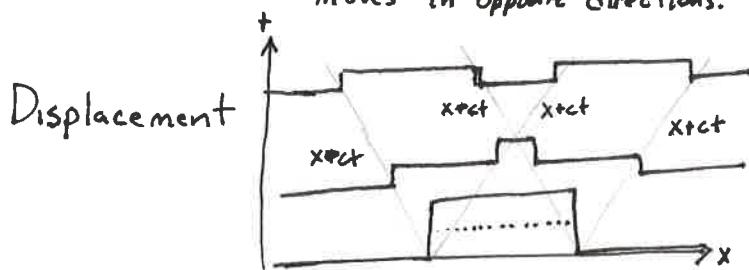
$$U_{tt} = c^2 U_{xx} \quad -\infty < x < \infty \quad 0 < t < \infty$$

$$U(x, 0) = f(x)$$

$$U_t(x, 0) = g(x)$$

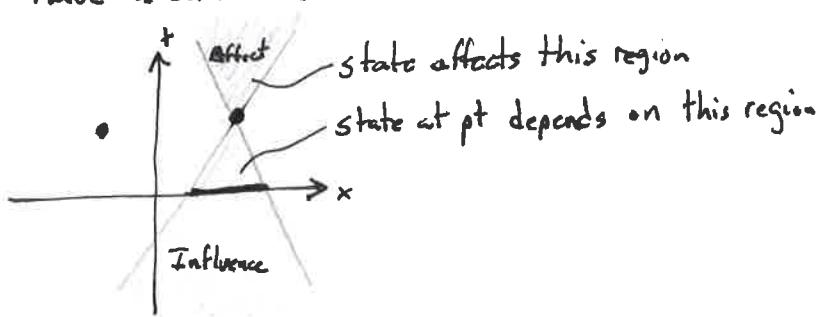
The general solution is

$$U(x, t) = \underbrace{\frac{1}{2} (f(x-ct) + f(x+ct))}_{\text{Split IC displacement into 2 equal halves. Each part moves in opposite directions.}} + \underbrace{\frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi}_{\text{Average of IC velocity } \cdot t \text{ within a domain of influence.}}$$



## Zone of Action

Wave equations have a domain of influence based on a finite wave speed.



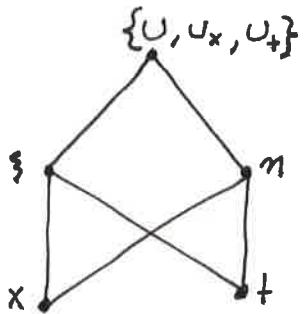
# D'Alembert Derivation

$$U_{tt} = c^2 U_{xx}$$

$$U(x, 0) = f(x)$$

$$U_t(x, 0) = g(x)$$

Pick new variables  $\xi = x + ct$  and  $\eta = x - ct$



$$U_x = U_\xi \cancel{\frac{\partial}{\partial x}} + U_\eta \cancel{\frac{\partial}{\partial x}} = U_\xi + U_\eta$$

$$U_{xx} = U_{x\xi} \cancel{\frac{\partial}{\partial x}} + U_{x\eta} \cancel{\frac{\partial}{\partial x}} = U_{x\xi} + U_{x\eta}$$

$$= (U_\xi + U_\eta)_\xi + (U_\xi + U_\eta)_\eta$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$U_t = U_\xi \cancel{\frac{\partial}{\partial t}} + U_\eta \cancel{\frac{\partial}{\partial t}}$$

$$= cU_\xi - cU_\eta$$

$$U_{tt} = U_{t\xi} \cancel{\frac{\partial}{\partial t}} + U_{t\eta} \cancel{\frac{\partial}{\partial t}}$$

$$= (cU_\xi - cU_\eta)_\xi c - (cU_\xi - cU_\eta)_\eta c$$

$$= c^2 (U_{\xi\xi} - U_{\xi\eta} - U_{\eta\xi} + U_{\eta\eta})$$

Substitute into governing

$$c^2 (U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}) = c^2 (U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta})$$

Reduce to

$$4U_{\xi\eta} = 0 \Rightarrow \boxed{U_{\xi\eta} = 0}$$

$\frac{d}{d\xi} \left( \frac{dU}{d\eta} \right)$

The solution along a characteristic does not change in magnitude.

Integrate wrt  $\eta$  or  $\xi$

$$\int U_{\xi\eta} d\eta = 0 \Rightarrow U_\eta = \text{function}(\xi)$$

Integrate wrt other coordinate

$$\int U_\eta d\eta = \int \text{function}(\xi) \Rightarrow U = \text{function}_1(\xi) + \text{function}_2(\eta)$$

$$\text{So, } U = f(\xi) + g(\eta) \Rightarrow U = f(x+ct) + g(x-ct)$$

Book uses  $\phi$  and  $\psi$ , good idea

$$U = \phi(\xi) + \psi(\eta) \Rightarrow U = \phi(x+ct) + \psi(x-ct)$$

Challenge:

Given an IC for displacement, find an IC for velocity such that only the right-traveling wave exists.

$$u(x,t) = \frac{1}{2} f(x+ct) + \underbrace{\frac{1}{2} f(x-ct)}_{\text{we only want this term.}} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi$$

Thus,  $\frac{1}{2} f(x+ct) = -\frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi$  Find  $g(\xi)$

Can you find a solution to this question?

# Derivation of D'Alembert (Hints)

General Solution

$$U(x,t) = \phi(x-ct) + \psi(x+ct) = \phi(\eta) + \psi(\xi)$$

Initial Conditions

$$U(x,0) = f(x) = \phi(x) + \psi(x) \quad \text{since } t=0$$

$$U_t(x,0) = g(x) = \frac{d\phi}{dt}(\eta) + \frac{d\psi}{dt}(\xi)$$

$$= \frac{d\phi}{d\eta} \frac{d\eta}{dt} + \frac{d\psi}{d\xi} \frac{d\xi}{dt} = \frac{d\phi}{d\eta} \left( \frac{d(x-ct)}{dt} \right) + \frac{d\psi}{d\xi} \left( \frac{d(x+ct)}{dt} \right)$$

$$= \frac{d\phi}{d\eta}(-c) + \frac{d\psi}{d\xi}(+c) \quad \text{but at } t=0, \eta=\xi$$

$$= -c \frac{d\phi}{d\eta} + c \frac{d\psi}{d\xi}$$

$$= -c \frac{d\phi}{dx} + c \frac{d\psi}{dx}$$

Now, 2 equations and 2 unknowns.

Integrate  $\int_{\eta_0}^{\eta} g(\eta) d\eta = \int_{\eta_0}^{\eta} -c \frac{d\phi}{d\eta} + c \frac{d\psi}{d\eta} d\eta$

$$\int_{\eta_0}^{\eta} g(\eta) d\eta = -c \phi(\eta) + c \psi(\eta) + \underbrace{c\phi(\eta_0) - c\psi(\eta_0)}_{\text{constants}}$$

Solve for  $\psi$

$$\psi = \frac{1}{2c} \int_{x_0}^{x'} g(\eta) d\eta + \frac{1}{2} \phi(x',t) - K \quad x' \text{ means } x+ct$$

Solve for  $\phi$

$$\phi = -\frac{1}{2c} \int_{x_0}^{x'} g(\eta) d\eta + \frac{1}{2} \psi(x',t) + K \quad x' \text{ means } x-ct$$

General solution

$$U(x,t) = \psi + \phi = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x_0}^{x+ct} g(\eta) d\eta - \frac{1}{2c} \int_{x'_0}^{x-ct} g(\eta) d\eta - K + K$$

$$U(x,t) = \frac{1}{2} \left( f(x+ct) + f(x-ct) \right) + \frac{1}{2c} \int_{x_0'}^{x+ct} g(\eta) d\eta - \frac{1}{2c} \int_{x_0'}^{x-ct} g(m) dm$$

+  $\frac{1}{2c} \int_{x-ct}^{x_0'} g(m) dm$

Combine the integral

$$\int_{x-ct}^{x_0'} + \int_{x_0'}^{x+ct} = \int_{x-ct}^{x+ct}$$

$$U(x,t) = \frac{1}{2} \left( f(x+ct) + f(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(m) dm$$