

GES 554

Lecture 11

15th Feb

2D Duct

Derive Acoustic Wave Equation

Continuity

$$\frac{d\rho}{dt} + \nabla \cdot (\rho V) = 0$$

linearize with no mean flow

$$\frac{d\rho'}{dt} + \rho_0 \nabla \cdot V' = 0$$

$$\begin{aligned} \rho &= \bar{\rho} + \rho' \\ u &= \bar{u} + u' \end{aligned}$$

Momentum

$$-\nabla p' = \rho \frac{dV'}{dt}$$

linearized with no mean flow

$$-\nabla p' = \rho \frac{dV'}{dt}$$

Energy

$$\rho \frac{De}{Dt} = -p \nabla \cdot V$$

linearize with ideal gas approximation and constant c_p

$$p' = a_0^2 \rho'$$

Apply ∇ to momentum

$$-\nabla^2 p' = \nabla \cdot \left(\rho \frac{dV'}{dt} \right)$$

Apply d/dt to continuity

$$\frac{d^2 \rho'}{dt^2} + \nabla \cdot \left(\rho \frac{dV'}{dt} \right) = 0$$

Substitute

$$-\nabla^2 p' = -\frac{d^2 p'}{dt^2}$$

Substitute

$$\boxed{\nabla^2 p' = -\frac{1}{a_0^2} \frac{d^2 p'}{dt^2}}$$

Comments on 2D Acoustics Wave Equation.

- Wave guide. (technical term)
 - EE power transmission
 - Radar applications
 - Jet engine intakes (cylindrical coordinates)
 - Hydraulic systems.

.....

- Cut-on frequency is where exponential decay transitions to propagation.

$$\omega = \left(\frac{a}{d}\right) n\pi$$

Acoustic Wave Propagation

We will not derive it, but the wave equation applies to acoustics

$$\frac{d^2 p'}{dt^2} = a_w^2 \left(\frac{d^2 p'}{dx^2} + \frac{d^2 p'}{dy^2} \right)$$

$$p = \bar{p} + p'$$

$$v = \bar{v} + v'$$

$$p = \bar{p} + p'$$

What are the boundary conditions?

No flow normal to a wall $\Rightarrow v = 0$

Look at momentum transport equation.

$$\frac{d(\rho v)}{dt} + \frac{d}{dy}(\rho v^2 + p) = 0$$

Substitute perturbations

$$\frac{d((\bar{p} + p')(v'))}{dt} + \frac{d}{dy}((\bar{p} + p')(v')^2 + \bar{p} + p') = 0$$

Reduce ($p'v'$ is small, $\frac{d\bar{p}}{dy}$ is zero)

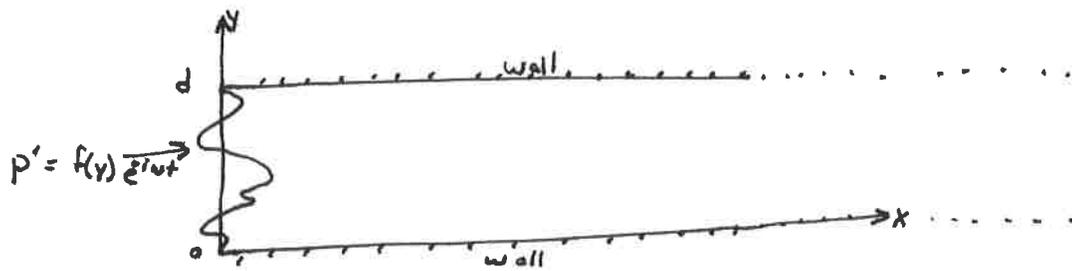
$$\frac{d(\bar{p} v')}{dt} + \frac{d}{dy}(p') = 0$$

$$\bar{p} \frac{dv'}{dt} + \frac{dp'}{dy} = 0$$

On a wall, v' is 0 $\Rightarrow \frac{dp'}{dy} = 0$

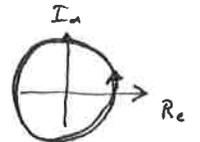
Note: this is how we could derive the governing...

2D Duct



Assume a solution and plug into Gov (Gou (wave eqn))

$$p'(x, y, t) = \tilde{p}(x, y) e^{-i\omega t}$$



$$\frac{dp'}{dt} = -i\omega \tilde{p}(x, y) e^{-i\omega t} \quad \frac{d^2 p'}{dx^2} = \frac{d^2 \tilde{p}}{dx^2} e^{-i\omega t} \quad \dots$$

$$i\omega^2 \tilde{p}(x, y) + \frac{\omega^2}{a_0^2} \left(\frac{d^2 \tilde{p}}{dx^2} + \frac{d^2 \tilde{p}}{dy^2} \right) = 0 \quad \Rightarrow \quad \boxed{\frac{\omega^2}{a_0^2} \tilde{p} + \frac{d^2 \tilde{p}}{dx^2} + \frac{d^2 \tilde{p}}{dy^2} = 0}$$

Separate the vars.

$$\tilde{p} = X(x) Y(y)$$

$$\frac{\omega^2}{a_0^2} X(x) Y(y) + X_{xx} Y + X Y_{yy} = 0$$

Divide by XY

$$\frac{\omega^2}{a_0^2} + \frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = 0 \quad \Rightarrow \quad \frac{Y_{yy}}{Y} = -\frac{X_{xx}}{X} - \frac{\omega^2}{a_0^2} = \alpha^2$$

Separated

$$Y_{yy} - \alpha^2 Y = 0$$

$$X_{xx} = \left(\frac{\omega^2}{a_0^2} - \alpha^2 \right) X$$

y solution

$$y = A \cos \alpha y + B \sin \alpha y$$

To enforce $\frac{dp'}{dy} = 0$ along walls, $y_y = 0 = -\alpha A \sin \alpha y + B \alpha \cos \alpha y$ at $y=0$ and $y=d$

$$0 = -\alpha A \sin \alpha d \quad \Rightarrow \quad \sin \alpha d = 0$$

$$\alpha = \frac{n\pi}{d}$$

$$\boxed{Y(y) = A \cos \frac{n\pi}{d} y}$$

X solution

$$X_{xx} = \left(\frac{\omega^2}{a_0^2} - \frac{n^2 \pi^2}{d^2} \right) X$$

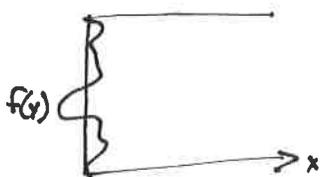
We don't restrict $\frac{\omega^2}{a_0^2} + \frac{n^2 \pi^2}{d^2}$ to be \pm , so ^{only one of}
use generic ~~form~~ form.
 e^{ikx}

$$X(x) = C e^{iDx} = C e^{\pm i \left(\frac{\omega^2}{a_0^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} x}$$

Total Solution

$$\tilde{p}(x,y) = A_n \cos\left(\frac{n\pi}{d}y\right) e^{\pm i \left(\frac{\omega^2}{a_0^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} x}$$

How do we find A_n ? Which BC does it represent?



~~At~~ At $x=0$, $\tilde{p} = A_n \cos \frac{n\pi}{d} y = f(y)$

mult by $\cos \frac{n\pi}{d} y$ and integrate.

$$\int_0^d f(y) \cos \frac{n\pi}{d} y dy = \int_0^d A_n \cos \frac{n\pi}{d} y \cos \frac{n\pi}{d} y dy$$

$$A_n = \frac{2}{d} \int_0^d f(y) \cos \frac{n\pi}{d} y dy$$

Completely Total solution

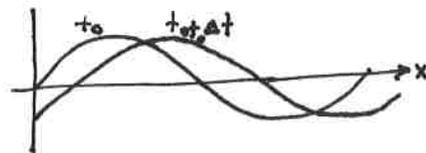
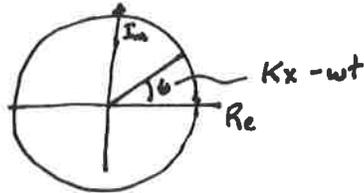
$$p'(x,y,t) = \underbrace{\frac{2}{d} \int_0^d f(y) \cos \frac{n\pi}{d} y dy}_{\text{Generating BC.}} \underbrace{\cos \frac{n\pi}{d} y}_{y\text{-dir}} \underbrace{e^{\pm i \left(\frac{\omega^2}{a_0^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} x}}_{x\text{-dir}} \underbrace{e^{-i\omega t}}_{\text{harmonic in time}}$$

Look at x-dir and harmonic time components

$$e^{\pm i \left(\frac{\omega^2}{a_0^2} \mp \frac{n^2 \pi^2}{d^2} \right)^{1/2} x - i \omega t}$$

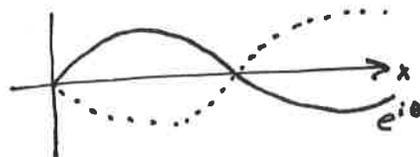
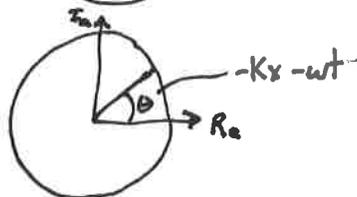
We have a \pm . (right running and left running waves)

1) $e^{+i(kx - \omega t)} = e^{i\theta}$



2) $e^{-i(kx + \omega t)} = e^{i\theta}$

$e^{+i(-kx - \omega t)} = e^{i\theta}$



Which one is left running and which is right running?

Look at x-dir coefficient

$$\left(\frac{\omega^2}{a_0^2} \mp \frac{n^2 \pi^2}{d^2} \right)^{1/2}$$

when $\frac{\omega^2}{a_0^2} > \frac{n^2 \pi^2}{d^2}$, the solution is $e^{i(kx - \omega t)}$

when $\frac{\omega^2}{a_0^2} < \frac{n^2 \pi^2}{d^2}$, the solution is $e^{i(iKx - \omega t)}$

$$= e^{-Kx - i\omega t}$$

$$= e^{-Kx} e^{-i\omega t}$$

The solution form depends on the frequency.

Small n propagates

Large n decays

alternative view

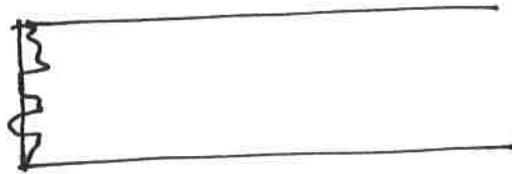
Large ω propagates

Small ω decays

Propagation vs. decay

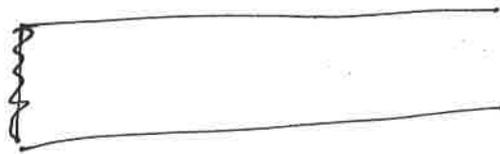
propagation

$$\frac{\omega^2}{a_0^2} > \frac{n^2 \pi^2}{d^2} \Rightarrow \omega^2 > a_0^2 \frac{n^2 \pi^2}{d^2} \Rightarrow \omega > \frac{a_0}{d} n \pi$$



Decay

$$\frac{\omega^2}{a_0^2} < \frac{n^2 \pi^2}{d^2} \Rightarrow \omega < \frac{a_0}{d} n \pi$$



Exactly $\frac{\omega^2}{a_0^2} - \frac{n^2 \pi^2}{d^2} = 0$

No x term in solution!

see demo.

Velocity computed from p' wave equation

$$\frac{d}{dt}(pU) + \frac{d}{dx}(pU^2 + p) = 0$$

$$\begin{aligned} p &= \bar{p} + p' \\ \rho &= \bar{\rho} + \rho' \\ U &= \bar{U} + U' \end{aligned}$$

$$\frac{d}{dt}((\bar{p} + p')(U')) + \frac{d}{dx}((\bar{p} + p')(U')^2 + \bar{p} + p') = 0$$

$$\bar{p} \frac{d}{dt}(U') + \frac{dp'}{dx} = 0 \quad \text{and by similarity} \quad \bar{p} \frac{d}{dt}(V') + \frac{dp'}{dy} = 0$$

Assume harmonic v'

$$v' = F e^{-i\omega t}$$

Find $\frac{dp'}{dx}$ and $\frac{dp'}{dy}$

$$\begin{aligned} \frac{dp'}{dx} &= A_n \cos \frac{n\pi}{d} y i \left(\frac{\omega^2}{a^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} e^{i \left(\frac{\omega^2}{a^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} x} e^{-i\omega t} \\ &= F e^{-i\omega t} \end{aligned}$$

Solve for F

$$F = A_n \cos \frac{n\pi}{d} y \left(\frac{\omega^2}{a^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} e^{i \left(\frac{\omega^2}{a^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2} x} (i)$$

Phase shift of (i) = 90° when $k = \left(\frac{\omega^2}{a^2} - \frac{n^2 \pi^2}{d^2} \right)^{1/2}$ is real
 i.e. $\omega > \frac{a}{d} n \pi$

Phase shift of $i^2 = -1 = 180^\circ$ when $\omega < \frac{a}{d} n \pi$