
GES 554

Lesson 12

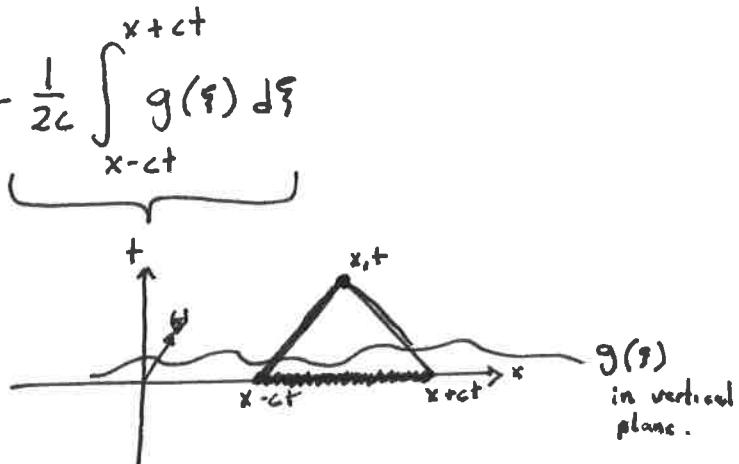
d' Alembert

D'Alembert Solution to the wave equation

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

or

$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi$$



The solution u consists of back tracked characteristics to the initial displacement BC and an average of the velocity BC.

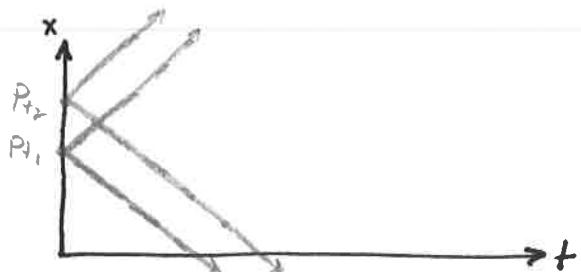
Why an ~~integral~~ at x,t ?

$$\text{Our WE is } u_{tt} = c^2 u_{xx}$$

Velocity is u_t , thus we can let $v = u_t$, thus the gov EQU becomes $v_t = c^2 u_{xx}$.

Wave Equation: D'Alembert Solutions

No walls

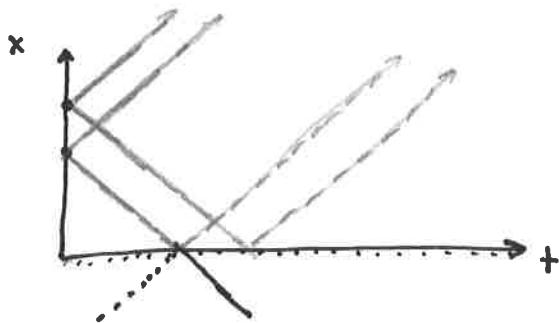


$$U_{tt} = U_{xx} \quad (c^2 = 1) \quad \text{and} \quad U_{\text{walls}} = 0$$

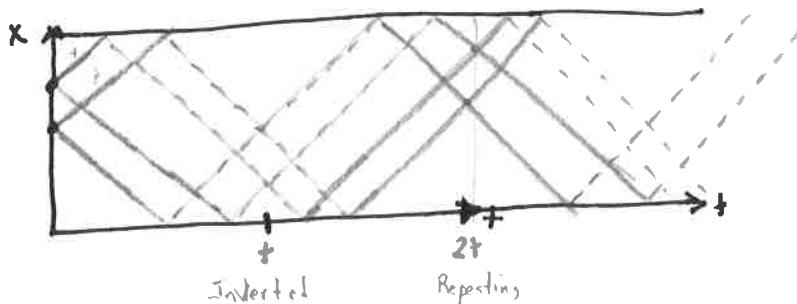
Characteristics are $x+t$ and $x-t$



One Wall



Two Walls



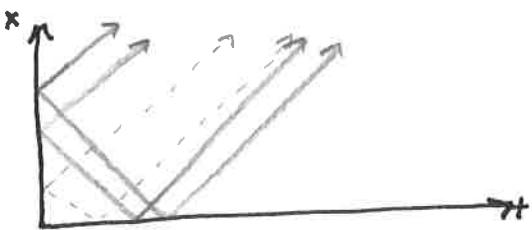
Water Wave / Acoustic Wave

$$P_{tt} = \alpha^2 \nabla^2 P' \quad \text{with} \quad \frac{dP'_{\text{wall}}}{dn_{\text{wall}}} = 0 = V'_{\text{wall}} = 0$$

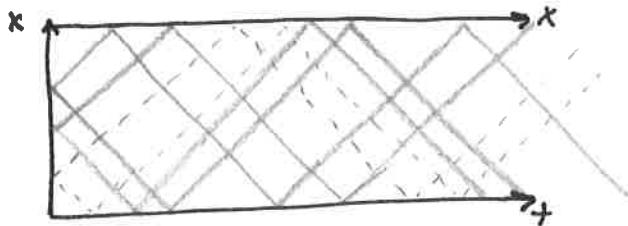
No Walls



One Wall



Two Walls



L18 P1

$$U_{tt} = U_{xx} \quad 0 < x < \infty \quad 0 < t < \infty$$

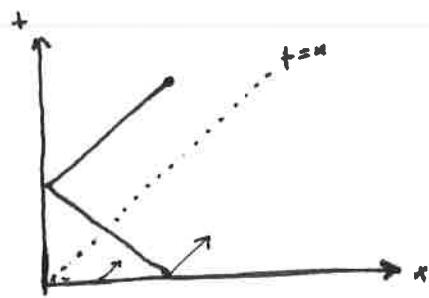
$$U(0, t) = 0$$

$$U(x, 0) = xe^{-x^2}$$

$$U_t(x, 0) = 0$$

Follow 18.8

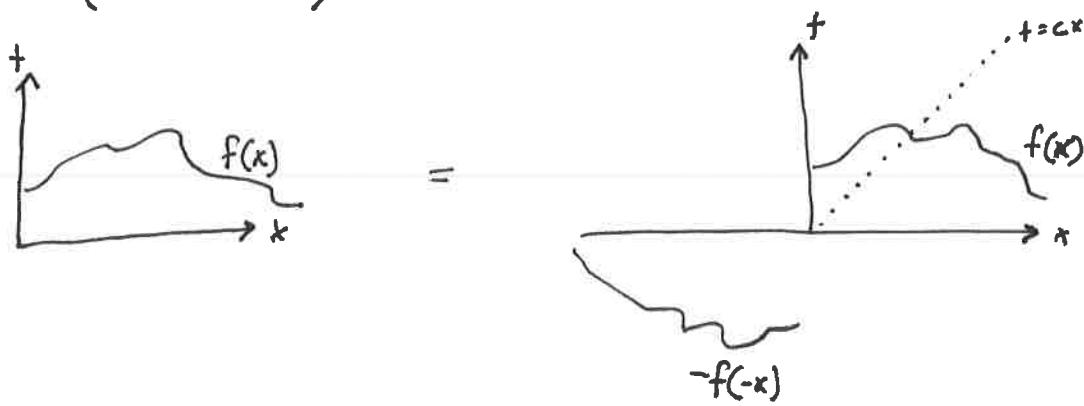
$$U(x, t) = \begin{cases} \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\alpha) d\alpha \\ \frac{1}{2} f(x + ct) - \frac{1}{2} f(x - ct) + \frac{1}{2c} \int_{ct-x}^{x+ct} g(\alpha) d\alpha & \text{when } x < ct \end{cases}$$



$$= \frac{1}{2} (x - ct) e^{-(x-ct)^2} + \frac{1}{2} (x + ct) e^{-(x+ct)^2} + 0 \quad \text{when } x > ct$$

$$= \frac{1}{2} (x + ct) e^{-(x+ct)^2} - \frac{1}{2} (t - x) e^{-(t-x)^2} \quad \text{when } x < t$$

L18 P2 (revision 1)



$$\begin{aligned}
 u(x,t) &= \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) - \frac{1}{2} f(ct-x) - \frac{1}{2} f(-ct-x) \\
 &= \underbrace{\frac{1}{2}(x-t) e^{-(x-t)^2}}_{\text{right}} + \underbrace{\frac{1}{2}(x+t) e^{-(x+t)^2}}_{\text{left}} - \underbrace{\frac{1}{2}(t-x) e^{-(t-x)^2}}_{\text{right}} - \underbrace{\frac{1}{2}(-t-x) e^{-(t+x)^2}}_{\text{left}} \\
 &= \begin{cases} \frac{1}{2}(x+t) e^{-(x+t)^2} + \frac{1}{2}(x-t) e^{-(x-t)^2} & \text{when } x > t \\ \frac{1}{2}(x+t) e^{-(x+t)^2} - \frac{1}{2}(t-x) e^{-(t-x)^2} & \text{when } 0 < x < t \end{cases}
 \end{aligned}$$

If you chose to do the problem this way, be careful.

For $x > t$

We chose the left and right waves of $f(x)$

For $0 < x < t$

We chose the left wave of $f(x)$ and the right of $-f(-x)$

BC at $x=0$

$$u = f(x) + (-f(-x)) = 0 \quad \text{always}$$

Boundary Conditions

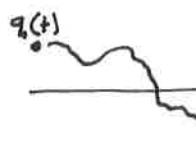
- Our D'Alembert solution has

implied BCs	$u(\infty)$ and $u(-\infty)$ are bounded
explicit ICs	$u(x,0)$ and $u_t(x,0)$

We need to consider other BC types. (as classified by Farlow)

- Controlled End Points.

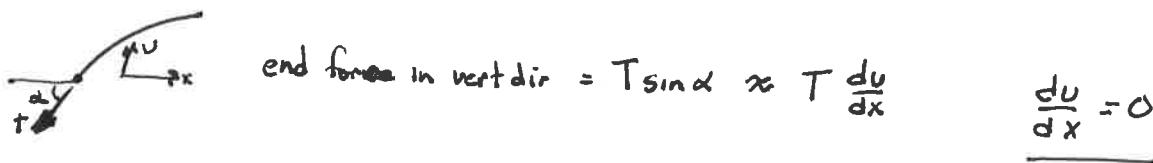
$$\begin{aligned} u(0,+) &= g_1(+) \\ u(1,+) &= g_2(+) \end{aligned}$$



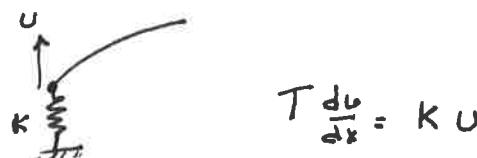
Clearly, these BCs are applied to non-infinite domains now.

- Force applied to End Points

Compare w Lecture 21



- Elastic Attachment to Boundaries



Drop a Slinky.

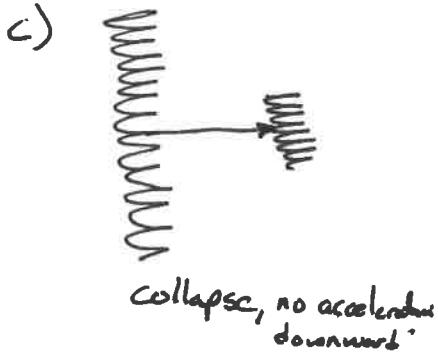
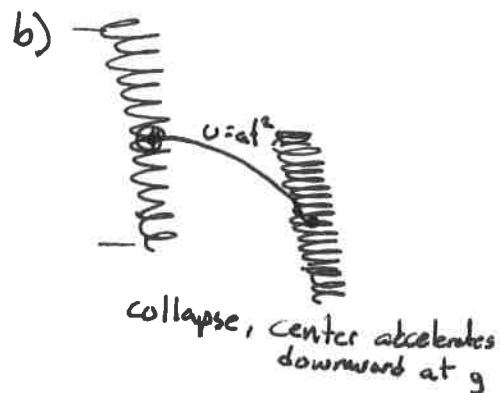
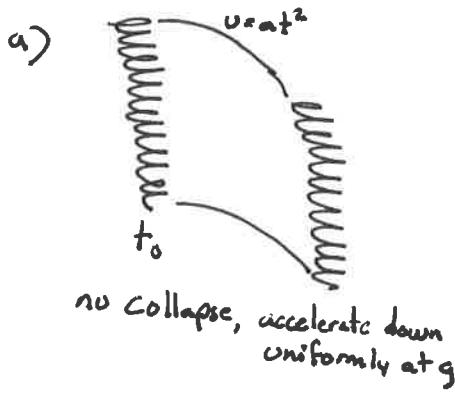


tiny.cc/GES554-SlinkySlow
tiny.cc/GES554-SlinkyIntro

Wave equation

$$\frac{d^2v}{dt^2} = \frac{E}{\rho} \frac{d^2v}{dx^2}$$

Hold a slinky by one end. Let the other end stretch downward.
Let go.... what happens?



d) Something else.



Comments:

- Rotational wave speed is faster than elastic wave speed.
- Everything behaves this way.
- Center of mass still behaves as a rigid body. CM is not at center of length!