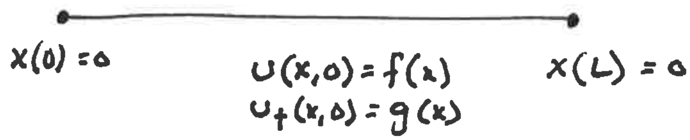
GES 554

Lesson 13

Wave Equations in Finite Domains

Wave Equation on finite domains.

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L$$



Separation of Vars.

$$u = X(x)\,T(t)$$

$$X T_{tt} = c^2 X_{xx} T$$

Collect terms

$$\frac{T_{tt}}{c^2 T} = \frac{X_{xx}}{X} = -\lambda^2$$

Separated PDEs

$$T_{tt} + c^2\lambda^2 T = 0 \qquad \Rightarrow \qquad T = A_1 \cos c\lambda t + B_1 \sin c\lambda t$$

$$X_{xx} + \lambda^2 X = 0 \qquad\qquad X = A_2 \cos \lambda x + B_2 \sin \lambda x$$

Apply BCs.

$$X(0) = 0 = A_2 \cancelto{1}{\cos 0} + B_2 \cancelto{0}{\sin 0} \quad \Rightarrow \quad A_2 = 0$$

$$X(L) = 0 = B_2 \sin \lambda L \quad \Rightarrow \quad \lambda L = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{L}$$

General Solution

$$u = \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos \frac{cn\pi}{L}t + b_n \sin \frac{cn\pi}{L}t\right)$$

Apply ICs $f(x)$

$$u(x,0) = a_n \sin \frac{n\pi}{L}x \cancelto{1}{\cos \frac{cn\pi}{L}0} + b_n \sin \frac{n\pi}{L}x \cancelto{0}{\sin \frac{cn\pi}{L}t} = f(x)$$

Integrate (mult by $\sin \frac{n\pi}{L}x$) to find $f(x)$ terms.

$$\int_0^L f(x) \sin \frac{n\pi}{L}x\,dx = \int_0^L a_n \sin^2 \frac{n\pi}{L}x\,dx \quad \Rightarrow \quad a_n = \frac{2}{L}\int f(x) \sin \frac{n\pi}{L}x\,dx$$

Alternative View

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Add these

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B + 0$$

Rearrange

$$\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

What if $A = \frac{n\pi x}{L}$ and $B = \frac{cn\pi t}{L}$

$$\sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L} = \frac{1}{2}\sin\left(\frac{n\pi}{L}(x+ct)\right) + \frac{1}{2}\sin\left(\frac{n\pi}{L}(x-ct)\right)$$

Q) Recognize this??!

A) Left and Right running traveling waves!

Standing waves are composed of traveling waves that cancel at nodes.

Apply ICs $g(x)$

$$U_t(x,0) = g(x) = -\sin\frac{n\pi}{L}x\left(a_n\frac{cn\pi}{L}\right)\cancelto{0}{\sin\frac{cn\pi}{L}t} + \sin\frac{n\pi}{L}x\left(b_n\frac{cn\pi}{L}\right)\cancelto{1}{\cos\frac{cn\pi}{L}t}$$

Mult by $\sin\frac{n\pi}{L}x$ and integrate.

$$\int_0^L \sin\frac{n\pi}{L}x\, g(x)\,dx = \int_0^L \sin^2\frac{n\pi}{L}x\; b_n\frac{cn\pi}{L}\,dx$$

$$b_n = \frac{L}{cn\pi}\frac{2}{L}\int_0^L \sin\frac{n\pi}{L}x\, g(x)\,dx$$

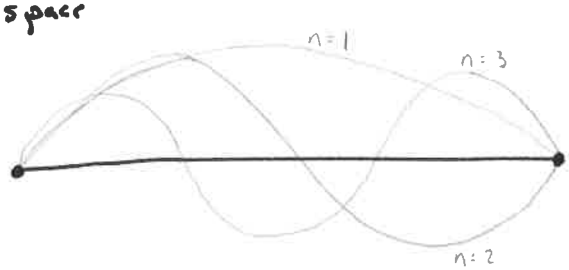
$$b_n = \frac{2}{cn\pi}\int_0^L \sin\frac{n\pi}{L}x\, g(x)\,dx$$

Generic Soln

$$\boxed{U(x,t) = \left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)\sin\frac{n\pi}{L}x\cos\frac{cn\pi}{L}t + \left(\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx\right)\sin\frac{n\pi x}{L}\sin\frac{cn\pi}{L}t}$$

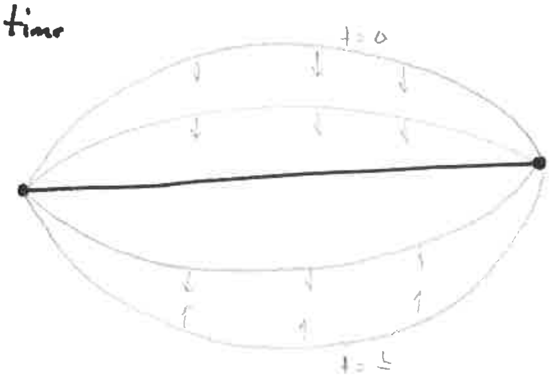
Harmonic

1) In space



Standing Wave

2) In time



L20 p1

$$U_{tt} = \alpha^2 U_{xx}$$

$$U(x,0) = \underbrace{\sin\left(\frac{x\pi}{L}\right)}_{n=1} + \underbrace{\frac{1}{2}\sin\left(3\pi\frac{x}{L}\right)}_{n=3}$$

Only IC for displacement.

$$U(x,t) = \sin\frac{\pi x}{L}\cos\frac{\alpha\pi t}{L} + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\alpha\frac{3\pi t}{L}\right)$$

L20 p2

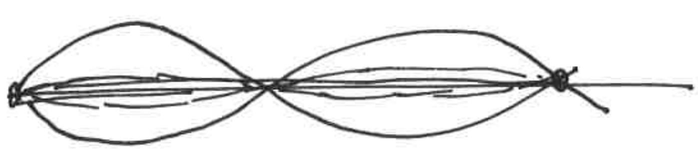
$$U_t(x,0) = \underbrace{\sin\left(3\frac{\pi x}{L}\right)}_{n=3}$$

Only IC for velocity.

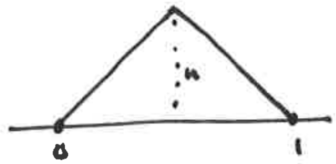
$$\boxed{U(x,t) = \sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi a t}{L}\right)\cdot\frac{L}{3\pi a}}$$

Watch out for this term! You can't read off a_n directly from the coefficient of ICs (for velocity).

$$U_{tt} = \alpha^2 U_{xx} - \beta U_t$$



L20P5 Guitar.



$$u(x,0) = \begin{cases} 2hx & 0 \le x \le \frac{1}{2} \\ 2h(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

Only displacement IC.

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad \boxed{\frac{n\pi}{L} = c}$$

$$= 2\int_0^{0.5} 2hx \sin \frac{n\pi x}{L}\, dx + 2\int_{0.5}^{1} 2h(1-x) \sin \frac{n\pi x}{L}\, dx$$

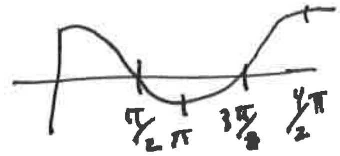
1^{st} term:

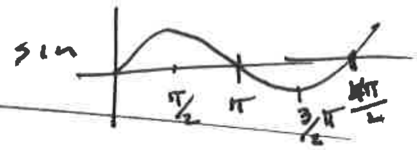
+-	F	G
+	$2hx$	$\sin cx$
−	$2h$	$-\frac{1}{c}\cos cx$
+	0	$-\frac{1}{c^2}\sin cx$

$$= -\frac{2hx}{c}\cos \overset{n\pi x}{\cancel{cx}} + \frac{2h}{c^2}\sin \overset{n\pi x}{\cancel{cx}} \Bigg|_0^{0.5}$$

$$\int_0^{0.5} x \sin(n\pi x)$$

$\pm$	F	G
+	x	$\sin(n\pi x)$
-	1	$\frac{-1}{n\pi}\cos(n\pi x)$
+	0	





$$\frac{\sin \frac{n}{2}\pi}{n^2\pi^2} - \frac{\cos\left(\frac{n\pi}{2}\right)}{2n\pi}$$

$$+ \frac{n\pi\cos\left(\frac{n\pi}{2}\right) + 2\sin\left(\frac{n\pi}{2}\right) - \cancel{\sin(n\pi)}^{0}}{2n^2\pi^2}$$

~~whatever~~:

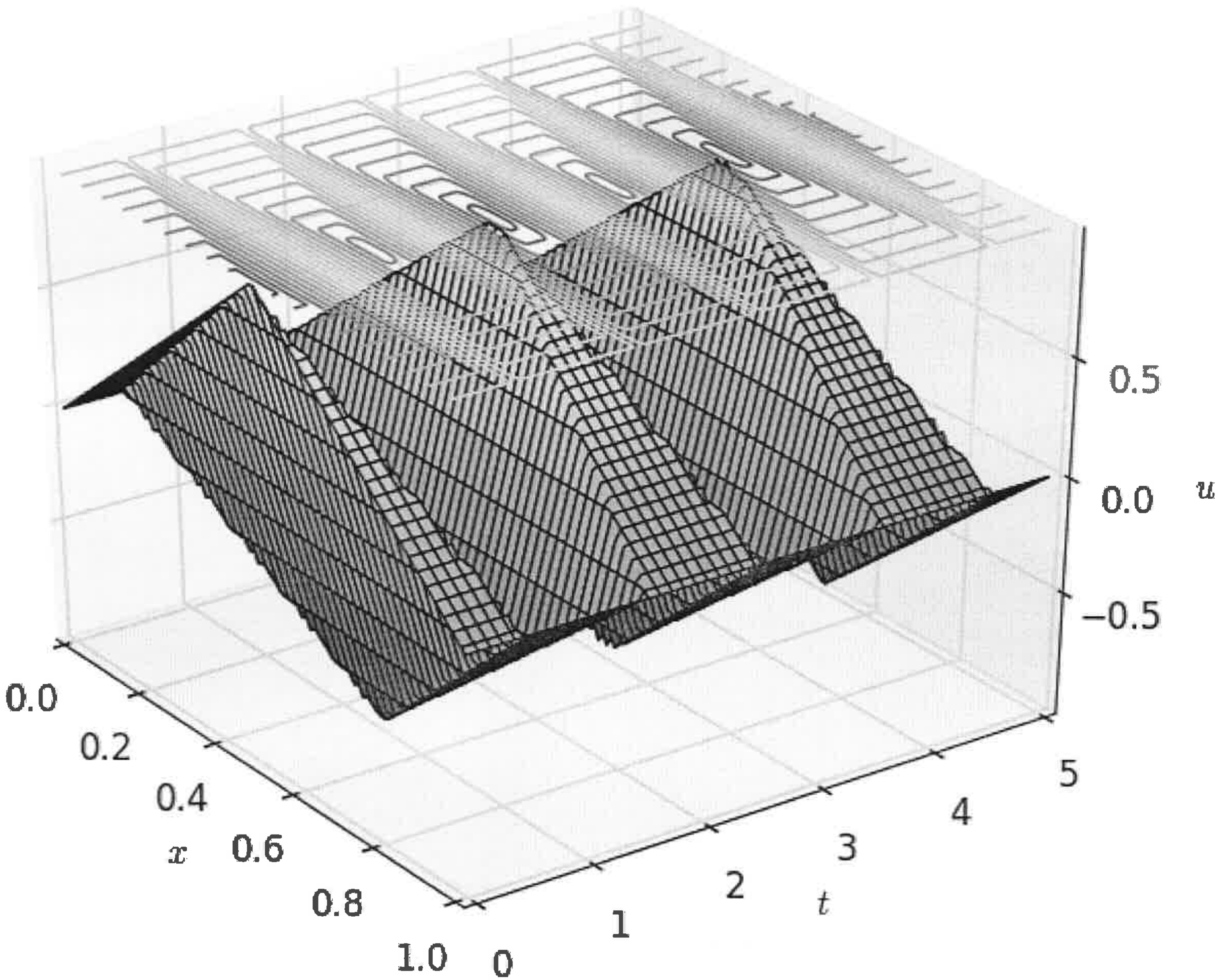
$n=1$ $\quad \frac{1}{\pi^2} - 0 + 0 + \frac{2 - }{2\pi^2} = \frac{1}{\pi^2} + \frac{1}{\pi^2} = \frac{2}{\pi^2}$

$n=2$ $\quad 0 - \frac{(-1)}{4\pi} + \frac{2\pi(-1) + 0}{2\cdot 4\cdot \pi^2} = \frac{1}{4\pi} - \frac{1}{4\pi} = 0$

$n=3$ $\quad \frac{-1}{9\pi^2} - 0 + \frac{\cancel{3\pi}^{0} - \cancel{2}}{\cancel{2}\cdot 9\pi^2} = -\frac{2}{9\pi^2}$

$n=4$ $\quad 0 - \frac{1}{2\cdot 4\cdot \pi} + \frac{\cancel{4\pi\cdot 1} + 0}{2\cdot \cancel{16\pi^2}\, 4\pi} = 0$

$n=5$


0.5
0.0
−0.5
u
0.0
0.2
0.4
0.6
0.8
1.0
x
0
1
2
3
4
5
t

Vibrating Beams.

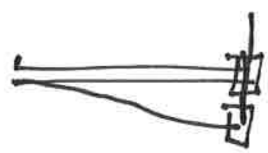
We already visited this topic during the project/labdemo.

$$U_{tt} + \sqrt{\frac{EI}{\eta}}\, U_{xxxx} = 0$$

Sepof Vars.

- Harmonic time terms $T(t) = A\cos t + B\sin t$
- 4 spatial terms $X(x) = A\sin x + B\cos x + C\sinh x + D\cosh x$

BCs.

- Free $U_{xx} = 0$, $U_{xxx} = 0$
- Fixed $U = 0$, $U_x = 0$ 
- Pinned $U = 0$, $U_{xx} = 0$ 
- Slider $U_x = 0$, $U_{xxx} = 0$ 

Each BC pair gives a different set of eigenvalues.