

GES 554

Lesson 14

Non dimensionalization

L20 p6

$$U_{tt} = \alpha^2 U_{xx} - \beta U_t$$

$$U(0,t) = 0$$

$$U(1,t) = 0$$

$$U(x,0) = f(x)$$

$$U_t(x,0) = 0$$

Try sep of vars

$$U = X T$$

$$X T_{tt} = \alpha^2 X_{xx} T - \beta X T_t$$

Collect like terms. (divide by XT)

$$\frac{T_{tt}}{T} = \alpha^2 \frac{X_{xx}}{X} - \beta \frac{T_t}{T}$$

$$\frac{T_{tt}}{\alpha^2 T} + \frac{\beta}{\alpha^2} \frac{T_t}{T} = \cancel{-} \frac{X_{xx}}{X} = -\lambda^2$$

ODE Gov Eqs

$$T_{tt} + \beta T_t + \lambda^2 \alpha^2 T = 0$$

$$X_{xx} + \lambda^2 X = 0$$

X solution

$$X = A \cos \lambda x + B \sin \lambda x \quad \text{Identical to non-damped solution}$$

We can find the B coefficients to fit $f(x)$. BCs imply $A=0$ for this case.
 $\lambda = n\pi$

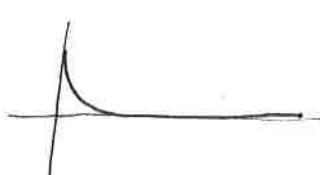
T solution

characteristic equation is $r^2 + \beta r + \lambda^2 \alpha^2 = 0$

$$r_1, r_2 = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\lambda^2 \alpha^2}$$

$$T(t) = A e^{-\frac{\beta}{2}t} e^{\frac{1}{2}\sqrt{\beta^2 - 4\lambda^2 \alpha^2} t} + B e^{-\frac{\beta}{2}t} e^{-\frac{1}{2}\sqrt{\beta^2 - 4\lambda^2 \alpha^2} t}$$

Thus T can be harmonic ($\beta=0$) or damped ($\beta > 0$) or divergent ($\beta < 0$)



Does β change the frequency? Yes

In the absence of damping, $\omega_n = \frac{1}{2}\sqrt{-4\lambda^2\alpha^2} = i\lambda\alpha$

So, rewrite as

$$T(t) = A e^{-\frac{\beta}{2}} e^{\frac{i}{2}\sqrt{\beta^2 + 4\omega_n^2}t} + B e^{-\frac{\beta}{2}} e^{-\frac{i}{2}\sqrt{\beta^2 + 4\omega_n^2}t}$$

- Step back to ODEs.

The canonical damped 2nd order ODE is

$$u_{tt} + \underbrace{2\zeta\omega_n u_t}_{\beta} + \underbrace{\omega_n^2 u}_{\lambda^2\alpha^2} = 0$$

Refer to your ODE class notes/book for a discussion of

- Overdamped ($\zeta > 1$) $u = a e^{r_1 t} + b e^{r_2 t}$



- Underdamped ($0 < \zeta < 1$) $u = a e^{r_1 t} + b e^{r_2 t}$



- Critically Damped ($\zeta = 1$) $u = a e^{r_1 t} + b t e^{r_1 t}$



Lesson 22 (Farlow) Dimensionless problems.

- Convert an engineering problem into a canonical math problem.

$$U_{tt} = \alpha U_{xx} + \beta U_{xx}$$

$$U(x, t) = \sin\left(\frac{\pi x}{L}\right)$$

- 1) Transform output to a dimensionless value (ie. divide by a reference solution value)
- 2) Remove constants in PDE by scaling the inputs.
- 2a) Convert/Transform ICs and BCs
- 3) Solve math problem
- 4) Transform back to engineering problem.

L22P1

Transform

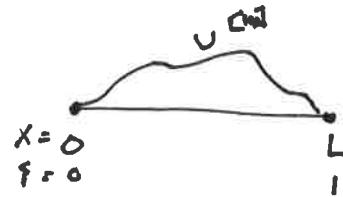
$$U_{tt} = \alpha^2 U_{xx} \quad 0 < x < L$$

$$U(0,t) = 0$$

$$U(L,t) = 0$$

$$U(x,0) = \sin\left(\frac{\pi x}{L}\right) + 0.5 \sin\left(\frac{3\pi x}{L}\right) \quad 0 < x < L$$

$$U_t(x,0) = 0$$



1) Output

$$U = \tilde{U} U_0 \quad \text{with } U_0 = 1 \quad \Rightarrow \text{no change.}$$

2) Inputs

$$x = \xi L \quad \text{and} \quad t = \left(\frac{L}{\alpha}\right) \tau = \frac{L}{\alpha} \tau \quad \Rightarrow \quad \tau = \tau^* = \frac{\alpha}{L} t$$

- time term

$$\frac{d^2 U^*}{d t^2} = \frac{d^2 \tilde{U}}{d \tau^{*2}} \quad \frac{d \tau^{*2}}{d t^2} = \left(\frac{\alpha^2}{L^2}\right) \frac{d^2 U^*}{d t^2} \quad \begin{matrix} \text{why 2? I prefer to use * "star" values.} \\ \text{You should use what makes sense to you.} \end{matrix}$$

- Space term

$$U_{xx} = \frac{d^2 U}{d x^2} = \frac{d^2 \tilde{U}}{d \xi^2} \frac{d \xi^2}{d x^2} = \frac{d^2 \tilde{U}}{d \xi^2} \left(\frac{1}{L}\right)^2$$

- Substitute.

$$\frac{\alpha^2}{L^2} U_{tt} = \left(\frac{1}{L^2}\right) \alpha^2 U_{xx} \quad \Rightarrow \quad \boxed{U_{tt} = U_{xx}}$$

3) BCs + ICs

BCs, no change

ICs.

$$U(x,0) \Rightarrow U\left(\frac{\xi}{L}, 0\right) = \sin\left(\frac{\pi \xi}{L}\right) + 0.5 \sin\left(\frac{3\pi \xi}{L}\right)$$

$$\boxed{U_{tt}(0,0) = 0}$$

$$= \boxed{\sin(\pi \xi) + 0.5 \sin(3\pi \xi)}$$

$$\begin{matrix} 0 < x < L \\ \hline L \\ = \\ 0 < \xi < 1 \end{matrix}$$

L22P2 Find a dimensionless form for

$$U_t = \alpha^2 U_{xx} \quad 0 < x < L$$

$$U(0,t) = T_1$$

$$U(L,t) = 0 \quad 0 < t < \infty$$

$$U(x,0) = T_2 \quad 0 \leq x \leq L$$

1) Define output in terms of a reference U value. T_1 sounds good.

$$U = U^* T_1 \Rightarrow U^* T_1 = \alpha^2 U_{xx}^* T_1 \Rightarrow U^* = \alpha^2 U_{xx}^*$$

2) Define inputs similarly. $U(0,t) = T_1 \Rightarrow U^* T_1 = T_1 \Rightarrow U^*(0,t) = 1$

$$U(x,0) = T_2 \Rightarrow U^* T_1 = T_2 \Rightarrow U^*(x,0) = T_2/T_1$$

$$t = t^* t_0 = T t_0 \quad x = x^* x_0 = \xi x_0$$

- $U^* = U_{tt}^* \frac{dt^*}{dt} = U_{tt}^* t_0 \quad \text{and} \quad U_{xx}^* = U_{x^* x^*}^* \frac{dx^*}{dx} = U_{\xi\xi}^* x_0^2$

- $U_{tt}^* t_0 = \alpha^2 U_{\xi\xi}^* x_0^2 \Rightarrow U_{tt}^* = \frac{\alpha^2 x_0^2}{t_0} U_{\xi\xi}^*$

We want $\frac{\alpha^2 x_0^2}{t_0} = 1$

- $0 < x < L \Rightarrow 0 = \xi x_0$

$$x = \xi x_0 \quad \Rightarrow \quad \boxed{\xi = \frac{x}{L}} \quad (\text{ie } x_0 = \frac{L}{\xi})$$

- $\frac{\alpha^2 x_0^2}{t_0} = 1 \Rightarrow \frac{\alpha^2 \left(\frac{L}{\xi}\right)^2}{t_0} = 1 \Rightarrow t_0 = \frac{\alpha^2 L^2}{\xi^2} \Rightarrow \boxed{T = \frac{L^2}{\alpha^2} \xi}$

3) ICs, BCs. Already transformed (no inputs)

4) Math problem

$$U_{tt}^* = U_{\xi\xi}^*$$

$$0 < \xi < 1$$

$$x = \xi L$$

$$U^*(0,\tau) = 1$$

$$0 < \tau < \infty$$

$$T = \frac{\alpha^2}{L^2} \tau$$

$$U^*(1,\tau) = 0$$

$$0 \leq \xi \leq 1$$

$$U(0,0) = \frac{T_2}{T_1}$$

L22 p5

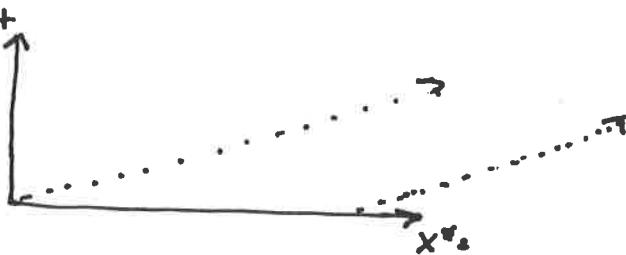
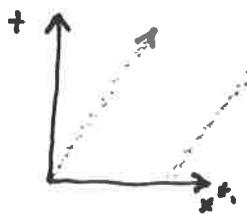
Pick ξ so that v is eliminated in

$$U_t + V U_x = 0$$

Solution:



Conceptually, we want to scale x such that v is 1.



2) $x = \xi x_0$

$$U_t + V \frac{dU}{d\xi} \frac{dx}{dx_0} = 0$$

$$U_t + V \frac{dU}{d\xi} \frac{1}{x_0} = 0 \Rightarrow x_0 = V$$

$$U_t + V U_\xi \frac{1}{V} = 0$$

$$\boxed{U_t + U_\xi = 0}$$

General 2nd Order ^{linear} PDE in x, y

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

- Hyperbolic ($B^2 - 4AC > 0$) Ex. wave eqn
- Parabolic ($B^2 - 4AC = 0$) Ex. heat equation
- Elliptic ($B^2 - 4AC < 0$) Ex. Laplace's equation

Note #1.

x and y are not restricted to space. y could be time. ($y=t$)

Note #2.

Does the type depend on D, E, F ? No

Examples.

This says that the heat equation is parabolic regardless of E term (U_t)

$$U_t - \alpha^2 U_{xx} = 0$$
$$A U_{xx} + B U_{xt} + C U_{yy} + D U_x + E U_t + F_u = G$$

$$B^2 - 4AC = 0 - 4A \cdot 0 = 0 \quad \text{parabolic.}$$

So,

$$U_t - \alpha^2 U_{xx} + D U_x + F_u = G$$

is also parabolic.

Laplace's eqn.

$$\nabla^2 U = 0 = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = U_{xx} + U_{yy}$$
$$A U_{xx} + B U_{xy} + C U_{yy} + D U_x + E U_y + F_u = G$$

$$B^2 - 4AC = 0 - 4 \cdot A \cdot C_+ < 0 \quad \underline{\text{Elliptic}}$$

Burgers' equation

$$U_t + 2U U_x = 0$$
$$A U_{xx} + B U_{xt} + C U_{tt} + D U_x + E U_t + F_u = G$$