

GES 554

Canonical Hyperbolic
+

2D \rightarrow 3D waves

L15P1

$$U_t = U_{xx} - 2U_x$$

$$U(x,0) = \sin(x)$$



Visually identify diffusion terms U_{xx} and convection terms $-2U_x$

Strategy: Transform from x coordinate to a new ξ coordinate traveling at velocity 2.

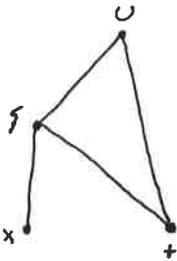
$$\xi = x - 2t$$

$\frac{du}{dt}$ doesn't change since only x is transformed, right? Charge ahead!

$$\frac{du}{dx} = \frac{du}{d\xi} \frac{d\xi}{dx} = \frac{du}{d\xi} \cdot 1 = \frac{du}{d\xi} \Rightarrow \frac{d^2u}{dx^2} = \frac{d^2u}{d\xi^2}$$

PDE: $\frac{du}{dt} = U_{\xi\xi} - 2U_{\xi}$ Nothing happened! What is wrong?

Try again, be consistent with transform.



$$\frac{du}{dx} = \frac{du}{d\xi} \frac{d\xi}{dx} \quad \text{and} \quad \frac{d^2u}{dx^2} = \frac{d^2u}{d\xi^2}$$

$$\frac{du}{dt} = \frac{du}{d\xi} \frac{d\xi}{dt} + \frac{du}{dt} = \frac{du}{d\xi} (-2) + \frac{du}{dt}$$

Notice this term!

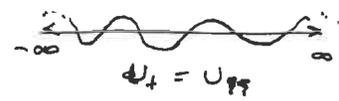
Subs into PDE

$$\frac{du}{dt} - 2 \frac{du}{d\xi} = \frac{d^2u}{d\xi^2} - 2U_x \Rightarrow \frac{du}{dt} = \frac{d^2u}{d\xi^2}$$

So in the ξ direction/coordinate, this is only a diffusion problem.

Use Fourier technique for $\xi = -\infty \leftrightarrow \infty$

$$F(u) = U \quad \Rightarrow \quad U(t) = e^{-\omega^2 t} U_0$$



and $F(IC) = F(\sin(x)) = \text{back at book} = i\sqrt{\frac{\pi}{2}} (\delta(\omega+1) - \delta(\omega-1))$

$$U = e^{-\omega^2 t} i\sqrt{\frac{\pi}{2}} (\delta(\omega+1) - \delta(\omega-1))$$

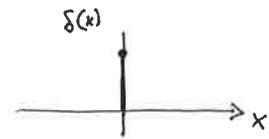
$U = F^{-1}(U)$ but \uparrow is certainly not in the book! So messy, right?!

Look at the definition of $F^{-1}(U) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega) e^{i\omega x} d\omega$

This would look like a nasty integral. But we have δ functions!!

$$U = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2 t} i\sqrt{\frac{\pi}{2}} (\delta(\omega+1) - \delta(\omega-1)) e^{i\omega x} d\omega$$

only has value when $\omega = -1$
only has value when $\omega = +1$



$$= \underbrace{\frac{i}{2} e^{-t} e^{-ix}}_{\omega = -1} - \underbrace{\frac{i}{2} e^{-t} e^{ix}}_{\omega = +1} = \frac{i}{2} e^{-t} (e^{-ix} - e^{ix}) \cdot \frac{i}{1} = \frac{e^{-t}}{2i} (e^{ix} - e^{-ix})$$

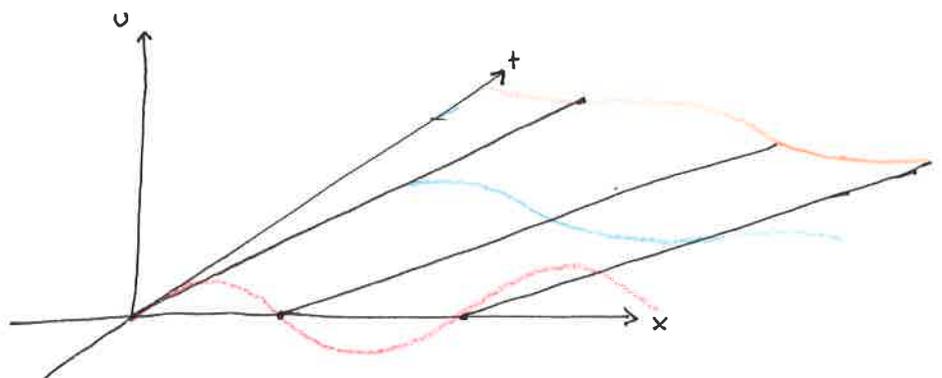
Recognize this?!

$$= e^{-t} \sin(x) \quad \leftarrow \text{technically, this is } \xi \quad = e^{-t} \sin(\xi)$$

Transform back to x coordinates $x = \xi + 2t \quad \Rightarrow \quad \xi = x - 2t$

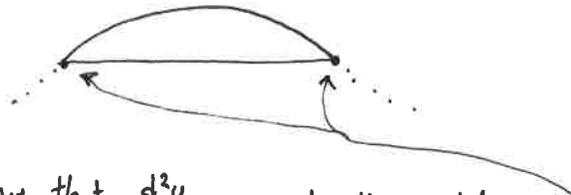
$$U = e^{-t} \sin(x - 2t)$$

Translating sine wave that decreases in magnitude in time.



Faster solution

Isolate one cell of sine wave



We know that $\frac{d^2 u}{d \xi^2} = 0$ at the nodal points

Thus, the solution should be the finite domain solution as well.

$$U(\xi, 0) = \sin(\xi) \quad 0 < \xi < \pi$$

Cell solution

$$U(x, t) = e^{-t} \sin(\xi) \quad \text{in } 0 < \xi < \pi$$

Tile solution (\pm) across domain.

$$U = e^{-t} \sin(\xi)$$

Transform $\xi \rightarrow x - 2t$

$$U = e^{-t} \sin(x - 2t)$$

L23P3 Verify Hyperbolic

$$3U_{xx} + 7U_{xy} + 2U_{yy} = 0$$

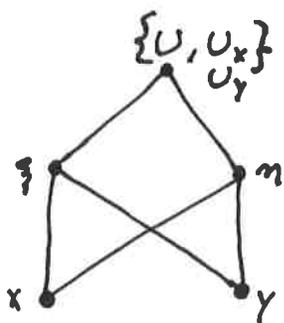
Verify:

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + Fu = G$$

$$B^2 - 4AC = 7^2 - 4 \cdot 3 \cdot 2 = 49 - 24 = 25 > 0$$

$25 > 0$
Hyperbolic

Find characteristic coordinate. (only $U_{\xi\eta} \neq 0$)



$$U_x = U_\xi \xi_x + U_\eta \eta_x$$

$$U_y = U_\xi \xi_y + U_\eta \eta_y$$

$$U_{xx} = U_{x\xi} \xi_x + U_{x\eta} \eta_x$$

$$= (U_{\xi\xi} \xi_x + U_{\eta\xi} \eta_x) \xi_x + (U_{\xi\eta} \xi_x + U_{\eta\eta} \eta_x) \eta_x$$

$$= U_{\xi\xi} \xi_x^2 + \underline{2U_{\xi\eta} \xi_x \eta_x} + U_{\eta\eta} \eta_x^2$$

$$U_{xy} = U_{x\xi} \xi_y + U_{x\eta} \eta_y$$

$$= (U_{\xi\xi} \xi_x + U_{\eta\xi} \eta_x) \xi_y + (U_{\xi\eta} \xi_x + U_{\eta\eta} \eta_x) \eta_y$$

$$= U_{\xi\xi} \xi_x \xi_y + \underline{U_{\eta\xi} \eta_x \xi_y + U_{\xi\eta} \xi_x \eta_y} + U_{\eta\eta} \eta_x \eta_y$$

$$U_{yy} = \text{similar to } U_{xx}$$

$$= U_{\xi\xi} \xi_y^2 + \underline{2U_{\eta\xi} \eta_y \xi_y} + U_{\eta\eta} \eta_y^2$$

Substitute into Gov Egu. and solve for \bar{b} values. (see Farlow p177)

We want only $\bar{B} \neq 0$, such that characteristics exist.

• Substitute and collect like terms

$$\bar{A} u_{\xi\xi} + \bar{B} u_{\xi\eta} + \bar{C} u_{\eta\eta} + \bar{D} u_{\xi} + \bar{E} u_{\eta} + \bar{F} u + \bar{G}$$

$$\bar{A} = A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2 =$$

$$\bar{B} = A 2 \eta_x \xi_x + B (\eta_x \xi_y + \xi_x \eta_y) + C \eta_y \xi_y$$

$$\bar{C} = A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2$$

$\bar{D}, \bar{E}, \bar{F}, \bar{G}$ ignore.

• Characteristics imply $\bar{A} = 0$
 $\bar{C} = 0$

$$A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2 = 0$$

$$A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2 = 0$$

$$\Rightarrow A \left(\frac{\xi_x}{\xi_y} \right)^2 + B \left(\frac{\xi_x}{\xi_y} \right) + C = 0$$

$$A \left(\frac{\eta_x}{\eta_y} \right)^2 + B \left(\frac{\eta_x}{\eta_y} \right) + C = 0$$

Solve for the $\left(\frac{\xi_x}{\xi_y} \right)$ and $\left(\frac{\eta_x}{\eta_y} \right)$ ratios that make $\bar{A} = \bar{C} = 0$

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{\eta_x}{\eta_y} = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A}$$

these can't be the same, so pick + and -

or - and +

but not + and +

or - and -

In fluids, $-\frac{\xi_x}{\xi_y}$ would be a streamline.

Return to our problem.

$$A = 3$$

$$B = 7$$

$$C = 2$$

$$\frac{m_x}{m_y} = \frac{-7 - \sqrt{49 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{-7 - 5}{6} = -\frac{12}{6} = -2$$

$$\frac{f_x}{f_y} = \frac{-7 + \sqrt{\dots}}{2 \cdot 3} = \frac{-7 + 5}{6} = -\frac{2}{6} = -\frac{1}{3}$$

Find η and ξ with x, y inputs.

$$\eta(x, y) = c_1 \text{ and } \xi(x, y) = c_2$$

~~Integration~~

$$d\eta = \eta_x dx + \eta_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

Integrate.

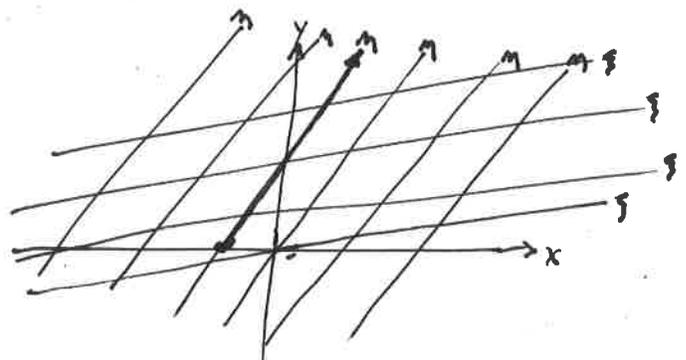
$$dy = -\frac{\eta_x}{\eta_y} dx = 2 dx \Rightarrow y = 2x + c_1 \Rightarrow \eta = c_1 = y - 2x$$

$$d\xi = \xi_x dx + \xi_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

Integrate

$$dy = -\frac{\xi_x}{\xi_y} dx = \frac{1}{3} dx \Rightarrow y = \frac{1}{3}x + c_2 \Rightarrow \xi = c_2 = y - \frac{1}{3}x$$

$$\eta(x, y) = y - 2x$$
$$\xi(x, y) = y - \frac{1}{3}x$$



L23P4.

$$U_{sm} = \Phi(\xi, \eta, u, v, w)$$

$$\bar{B} = 2A \eta_x \xi_x + B(\eta_x \xi_y + \xi_x \eta_y) + C \eta_y \xi_y$$

$$\bar{B} U_{sm} = 0 \quad \text{with} \quad \eta = y - 2x \quad \eta_x = -2 \quad \eta_y = 1$$

$$\xi = y - \frac{1}{3}x$$

$$\xi_x = -\frac{1}{3} \quad \xi_y = 1$$

$$\bar{B} = 2 \cdot 3 \cdot (-2) \cdot (-\frac{1}{3}) + 7((-2)(1) + (-\frac{1}{3})(1)) + 2(1)(1)$$

$$= 4 + -\frac{49}{3} + 2 = \text{some \#}$$

non-zero

Divide by \bar{B} to give

$$U_{sm} = 0$$

$$\Phi = 0$$

3D waves.

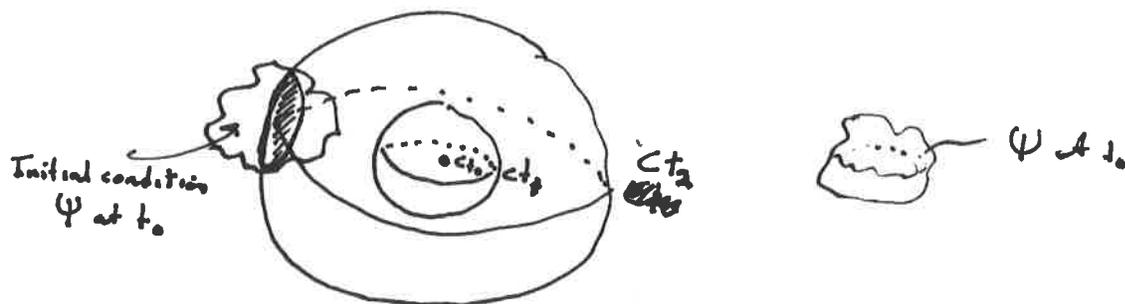
$$U_{++} = c^2 \nabla^2 U$$

$$U(x, y, z, 0) = \Phi(x, y, z)$$

$$U_t(x, y, z, 0) = \Psi(x, y, z)$$

We know that the $\left\{ \begin{array}{l} \text{characteristic} \\ \text{propagation} \end{array} \right\}$ speed is c .

A point at x, y, z and time c , $\left\{ \begin{array}{l} \text{sees} \\ \text{hears} \end{array} \right\}$ a sphere of radius ct .



Solution to Ψ (Farlow 184)

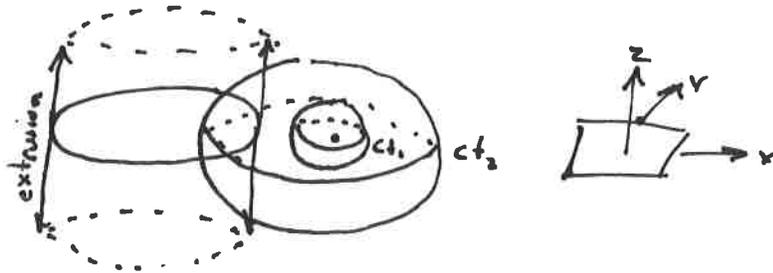
$$U(x, y, z, t) = t \bar{\Psi} = t \cdot \text{Average of } \Psi(r=ct) \text{ on shell (surface) of sphere.}$$

Note:

Once the sphere's surface passes an IC, there is no further contribution from that IC.

Method of Descent (Hadamard's) for 2D wave equations

- Keep 3D expanding sphere about x, y, z .
- Extrude 2D ICs into 3D



- $\bar{\Psi}$ occurs where the sphere and the extruded IC intersect.

Demo.

Note: Once the sphere's surface intersects the IC, the contribution is always felt into the future.