

GES 554

Systems of PDEs

# Systems of PDEs.

Up to now, we studied PDEs of one state,  $u$ .

$$u_t + f_x = 0$$

$$u_{tt} \neq u_{xx}$$

$$u_t + u_{xx} = 0$$

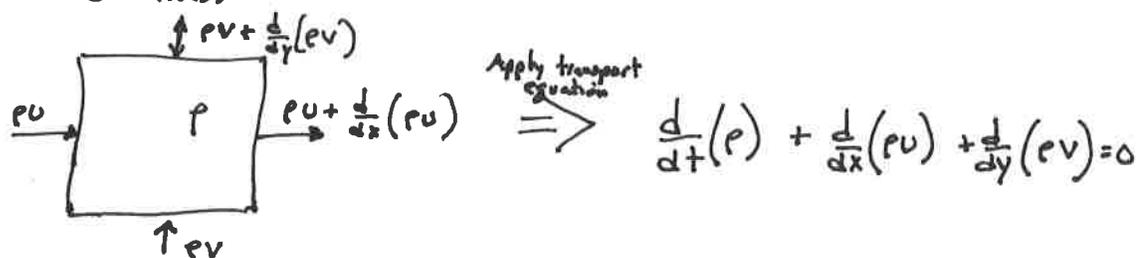
These were derived (usually) from a conservation equation(s) and simplified to one state.

Most physical systems/phenomena are a set of coupled states of PDEs.

## • Fluid flow.

- Conserve mass, momentum (multiple directions), energy, ...
- These conservation PDEs are coupled (often non-linearly)

### • Conservation of mass



### • Conservation of momentum

- x dir  $\frac{d}{dt}(\rho u) + \frac{d}{dx}(\rho u^2 + p) + \frac{d}{dy}(\rho u v) = \frac{d}{dx}(\tau_{xx}) + \frac{d}{dy}(\tau_{xy})$
- y dir ...
- z dir ...

### • Conservation of energy

$$\frac{d}{dt}(\rho h) + \frac{d}{dx}(\rho u h) + \frac{d}{dy}(\rho v h) = \frac{d}{dx}(u \tau_{xx} + v \tau_{xy} - q_x) + \frac{d}{dy}(u \tau_{xy} + v \tau_{yy} - q_y)$$

This is the Navier-Stokes equation

# Brief review of eigenvalues and eigenvectors of a matrix $A$

- eigenvalues

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} a_{11} & \dots & \dots \\ \vdots & \ddots & \vdots \\ \dots & \dots & a_{nn} \end{bmatrix}$$

- eigenvectors

$$v_i \text{ such that } Av_i = \lambda_i v_i$$

Example.

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$$

- singular?  $\det(A) = |A| = 0 \cdot 0 - 8 \cdot 2 = -16 \neq 0$  Not singular  $A^{-1}$  exists.

- eigenvalues

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 8 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 16 = 0 \Rightarrow \lambda = \sqrt{16} = \pm 4$$

$$\boxed{\text{eigv} = (+4, -4)}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

- eigenvectors

$$Av_1 = 4v_1 \quad \text{and} \quad Av_2 = -4v_2$$

$$1) \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} v_1 = 4v_1 \Rightarrow \begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{pick } v_1 = \begin{pmatrix} 1 \\ a \end{pmatrix} \text{ and solve}$$

$$-4 \cdot 1 + 8a = 0 \Rightarrow a = \frac{1}{2} \Rightarrow v_1 = \begin{pmatrix} 1 \\ .5 \end{pmatrix}$$

← singular, so don't even try to invert!!  
Not that it would do any good

$$2) \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} v_2 = -4v_2 \Rightarrow \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4 + 8a = 0 \Rightarrow a = -\frac{1}{2}$$

$$v_2 = \begin{pmatrix} 1 \\ -.5 \end{pmatrix}$$

$$\boxed{\text{eigvec} = \begin{pmatrix} 1 & 1 \\ .5 & -.5 \end{pmatrix}} = P$$

$$P^{-1} = \begin{bmatrix} -.5 & -1 \\ -.5 & 1 \end{bmatrix} = \begin{bmatrix} .5 & 1 \\ .5 & -1 \end{bmatrix}$$

- Verify  $P^{-1}AP = \Lambda$

# Forlow Lesson 29 Example

$$\frac{du_1}{dt} + 8 \frac{du_2}{dx} = 0$$

$$\frac{du_2}{dt} + 2 \frac{du_1}{dx} = 0$$

$$\Rightarrow \frac{dU}{dt} + A \frac{dU}{dx} = 0 \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$$

We can apply the characteristics concept to multiple dimensions.

Previously, we said  $Au_t + Bu_x \Rightarrow x(s), t(s) \Rightarrow Au_t + Bu_x = U_s$

Now, we say  $U_t + AU_x = 0 \Rightarrow \text{~~u_t + au_x = 0~~} \Rightarrow V_t + \Lambda V_x = 0$   
 $U = P(V)$  ↑ diagonal matrix.

- $U_t + AU_x = 0$

- Substitute  $U = P V \Rightarrow P V_t + A P V_x = 0$

- Left multiply by  $P^{-1}$  (second families?)  $\Rightarrow \underbrace{P^{-1} P}_{\text{Identity}} V_t + P^{-1} A P V_x = 0$

- Gives  $V_t + \underbrace{P^{-1} A P}_{\Lambda} V_x = 0$   
 $\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$

The multidimensional PDE is now decoupled in space  $V$ .

$$V_{1,t} + 4 V_{1,x} = 0$$

$$V_{2,t} - 4 V_{2,x} = 0$$

Solutions are:

$$V_1 = \phi(x - 4t)$$

$$V_2 = \psi(x + 4t)$$

L29 p3

$$\frac{du_1}{dt} + \frac{du_1}{dx} + \frac{du_2}{dx} = 0$$

$$\frac{du_2}{dt} + 4 \frac{du_1}{dx} + \frac{du_2}{dx} = 0$$

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$U_t + AU_x = 0$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$\det(A) = 1 - 4 = -3 \neq 0 \quad A^{-1} \text{ exists}$$

Concept Q): What if  $\det(A) = 0$ ? What can we do?

• Eig(A)

$$|\lambda I - A| = 0 = |A - \lambda I| \Rightarrow \begin{vmatrix} \lambda - 1 & -1 \\ -4 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2$$

$$\lambda = \{-1, 3\}$$

• Eigvec(A)

$$\lambda = -1 \quad \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2 - a = 0 \Rightarrow a = -2$$

$$\lambda = 3 \quad \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2 - a = 0 \Rightarrow a = 2$$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

$\lambda = -1 \quad \lambda = 3$

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ +2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

• Write PDE in v space  $U = Pv$  and  $v = P^{-1}U$

$$v(x,t) = \frac{1}{2} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}f(x) - \frac{1}{4}g(x) \\ \frac{1}{2}f(x) + \frac{1}{4}g(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \text{ for } v_1 \\ \psi(x) \text{ for } v_2 \end{pmatrix}$$

$$v_t + \Lambda v_x = 0 \Rightarrow \begin{cases} v_{2t} - v_{1x} = 0 \\ v_{2t} + 3v_{2x} = 0 \end{cases} \Rightarrow \begin{cases} v_1 = \phi(x+t) \\ v_2 = \psi(x+3t) \end{cases}$$

• General solution

$$U = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} v = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{2}f(x+t) - \frac{1}{4}g(x+t) \\ \frac{1}{2}f(x+3t) + \frac{1}{4}g(x+3t) \end{pmatrix} = \dots$$

Initial Conditions

$$\begin{aligned} u_1(x,0) = f(x) \\ u_2(x,0) = g(x) \end{aligned} = P v \quad \Rightarrow \quad v(x,0) = P^{-1} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} .5 & 1 \\ .5 & -1 \end{bmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$

$$v_1(x,0) = .5 f(x) + g(x) = \phi(x)$$

$$v_2(x,0) = .5 f(x) - g(x) = \psi(x)$$

General solution

$$u = P v = \begin{pmatrix} 1 & 1 \\ .5 & -.5 \end{pmatrix} v = \begin{aligned} &\phi(x-4t) + \psi(x+4t) \\ &\frac{1}{2} \phi(x-4t) - \frac{1}{2} \psi(x+4t) \end{aligned}$$

$$= \frac{1}{2} f(x-4t) + g(x-4t) + \frac{1}{2} f(x+4t) - g(x+4t)$$

$$\frac{1}{4} f(x-4t) + \frac{1}{2} g(x-4t) - \frac{1}{4} f(x+4t) + \frac{1}{2} g(x+4t)$$

As with the wave equation, this is valid for all  $f(x), g(x)$  !!

# Navier Stokes

multi dimensional transport eqn.

$$\frac{dU}{dt} + \nabla F = 0$$

$$U_{\geq 0} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \end{pmatrix}$$

$$\nabla F = \frac{dF_x}{dx} + \frac{dF_y}{dy}$$

$$F = F_{\text{inviscid}}^I - F_{\text{viscid}}^V$$

$$F_x^I = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uh \end{pmatrix}$$

$$F_y^I = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vh \end{pmatrix}$$

$$F_x^V = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} - \delta_x \end{pmatrix}$$

$$F_y^V = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} - \delta_y \end{pmatrix}$$

Can you prove these two conditions? (\$1M prize)

$$1) \vec{V}(x,t) \in [C^\infty(\mathbb{R}^3 \times [0, \infty))]^3, \quad p(x,t) \in C^\infty(\mathbb{R}^3 \times [0, \infty])$$

translation: velocity and pressure are smooth and continuous.

$$2) \int_{\mathbb{R}} |V(x,t)|^2 dx < E \text{ for all } t \geq 0$$

translation: kinetic energy is bounded.

When we drop the viscous terms, we obtain the Euler Equation

$$\begin{aligned} \tau = 0 \\ \delta = 0 \end{aligned} \Rightarrow F^V = 0$$

The stress  $\tau$  is often modeled with a Newtonian Fluid approximation.

$$\tau_{ij} = \mu \left( \frac{du_i}{dx_j} + \frac{du_j}{dx_i} \right) + \delta_{ij} \lambda \left( \frac{du_k}{dx_k} \right)$$

Ideal Gas

$$P = \rho R T = (\gamma - 1) \left( \rho e - \frac{1}{2} \rho \vec{V}^2 \right)$$

# Analysis of the Navier Stokes equation.

• Non-dimensional form of  $x$ -momentum

$$\rho \left( \frac{du_i}{dx_i} + \frac{du_j}{dx_j} \right) + \delta_{ij} \lambda \left( \frac{du_k}{dx_k} \right)$$

$$\frac{d(\rho u)}{dt} + \frac{d}{dx}(\rho u^2 + p) + \frac{d}{dy}(\rho v u) = \frac{d}{dx}(\tau_{xx}) + \frac{d}{dy}(\tau_{xy})$$

$$\begin{cases} \rho = \rho_0 \rho^* & u = U_0 u^* & v = U_0 v^* & t = t^* \frac{L_0}{U_0} & x = L_0 x^* \\ p = \rho_0 U_0^2 p^* & \mu = \mu_0 \mu^* & & & \end{cases}$$

Substitute

$$\frac{d(\rho_0 U_0 \rho^* u^*)}{dt^*} \frac{U_0}{L_0} + \frac{1}{L_0} \frac{d}{dx^*} (\rho_0 U_0^2 \rho^* u^{*2} + \rho_0 U_0^2 p^*) + \frac{1}{L_0} \frac{d}{dy^*} (\rho_0 U_0 U_0 \rho^* u^* v^*)$$

$$= \frac{1}{L_0} \frac{d}{dx^*} \left( \rho_0 \mu^* \left[ \frac{d(\rho_0 U_0 u^*)}{d(L_0 x^*)} + \dots \right] \right) + \frac{1}{L_0} \frac{d}{dy^*} (\dots)$$

only derivatives here  
so pick 1. The others  
are identical.

Same as  $\frac{d}{dx}$  term

Consolidate

$$\frac{\rho_0 U_0^2}{L_0} \frac{d(\rho^* u^*)}{dt^*} + \frac{\rho_0 U_0^2}{L_0} \frac{d}{dx^*} (\rho^* u^{*2} + p^*) + \frac{\rho_0 U_0^2}{L_0} \frac{d}{dy^*} (\rho^* u^* v^*) =$$

$$+ \frac{\rho_0 U_0}{L_0^2} \frac{d}{dx^*} \left( \mu^* \frac{d(u^*)}{dx^*} + \dots \right)$$

Divide by  $\frac{\rho_0 U_0^2}{L_0}$

$$\frac{d}{dt^*} (\rho^* u^*) + \frac{d}{dx^*} (\rho^* u^{*2} + p^*) + \frac{d}{dy^*} (\rho^* u^* v^*) = \frac{\mu_0 U_0}{L_0^2 \rho_0 U_0^2} \frac{d}{dx^*} (\mu^* \frac{du^*}{dx^*} \dots)$$

$$= \frac{1}{\left( \frac{\rho_0 U_0 L_0}{\mu_0} \right)} \frac{d}{dx^*} (\dots)$$

Q) Why should you non-dimensionalize complex PDEs?

Concept question?

What is this? Reynolds #

$$Re \equiv \frac{\rho U L}{\mu}$$

Analysis of the Euler equation. (ie.  $Re \gg 1$ ) in 1D  
 $\frac{1}{Re} \approx 0$

$$\frac{dU}{dt} + \nabla F = 0 \quad U = \begin{pmatrix} p \\ \rho u \\ \rho e \end{pmatrix} = \begin{pmatrix} p \\ m \\ E \end{pmatrix} \quad F_x = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u h \end{pmatrix} \rightarrow$$

Can we find characteristics? Yes (Ref. Numerical methods for Conservation Laws.)  
 LeVeque 1992

$$\frac{dF}{dU} = A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & (\gamma-1) \\ \frac{1}{2}(\gamma-1)u^3 - u \frac{E+p}{\rho} & \frac{E+p}{\rho} - (\gamma-1)u^2 & \gamma u \end{bmatrix} \quad \frac{dU}{dt} + A \frac{dU}{dx} = 0$$

Eigenvalues

$$\lambda = \{U-a, U, U+a\} \quad \text{where } a \text{ is the speed of sound}$$

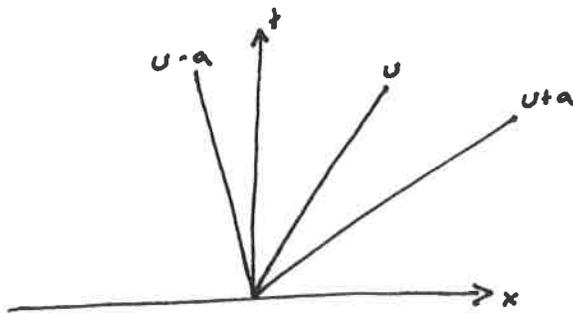
$$a = \sqrt{\gamma R T}$$

Eigenvectors

$$R = \begin{bmatrix} 1 & 1 & 1 \\ U-a & U & U+a \\ H-ua & \frac{1}{2}U^2 & H+ua \end{bmatrix}$$

$\lambda = U-a \quad \lambda = U \quad \lambda = U+a$

Euler's equation has 3 characteristics



See Sod Shock Tube