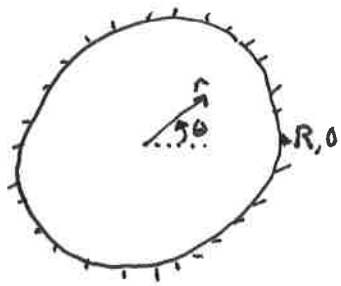
Drum head (Bessel's Equ)

Wave equation on polar domain



$$\underbrace{U_{tt} = c^2 \nabla^2 U}_{\text{wave equation}}$$

$$\nabla^2 U = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta}$$

Initial conditions

$$U = f(r,\theta)$$
$$U_t = g(r,\theta)$$

Boundary Condition

$$U(r=R) = 0$$

Sep of Vars (differs from Farlow slightly. He finds the Helmholtz equation.)

$$U = R(r)\,\Theta(\theta)\,T(t)$$

- pull off time term first

$$R(r)\Theta(\theta)\,T_{tt}(t) = c^2\left(R_{rr}(r)\,\Theta(\theta)T(t) + \frac{1}{r}R_r\,\Theta(\theta)T(t) + \frac{1}{r^2}R\,\Theta_{\theta\theta}T\right)$$

$$\frac{T_{tt}}{c^2 T} = \frac{R_{rr}\,\Theta}{R\Theta} + \frac{1}{r}\frac{R_r\,\Theta}{R\,\Theta} + \frac{1}{r^2}\frac{R\,\Theta_{\theta\theta}}{R\Theta} = -\lambda^2$$

$$\boxed{T_{tt} + \lambda^2 c^2 T = 0}$$ Harmonic: sine and cosine of time

- Space terms R, Θ (premultiply by $R r^2 \Theta$) (conc terms)

$$\Theta r^2 R_{rr} + \Theta r R_r + R\,\Theta_{\theta\theta} = -\lambda^2 R r^2 \Theta$$

Separate R and Θ (divide by ΘR

$$\frac{r^2 R_{rr}}{R} + \frac{r R_r}{R} + \frac{\Theta_{\theta\theta}}{\Theta} = -\lambda^2 r^2$$

$$\frac{r^2 R_{rr}}{R} + \frac{r R_r}{R} + \lambda^2 r^2 = -\frac{\Theta_{\theta\theta}}{\Theta} = +n^2$$

Θ term:

$$\boxed{\Theta_{\theta\theta} + \Theta n^2 = 0}$$

Harmonic in θ: sine and cosine of θ

R.term:

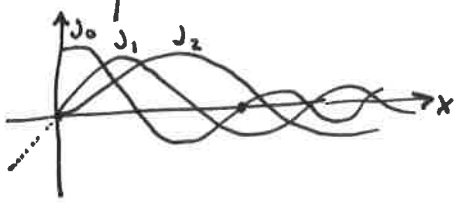
$$r^2 R_{rr} + r R_r + \lambda^2 r^2 R - R n^2 = 0$$

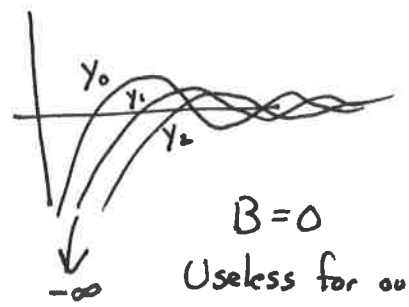
$$\boxed{r^2 R_{rr} + r R_r + (\lambda^2 r^2 - n^2) R = 0}$$

Bessel ~~Function~~ Equation

$$R(r) = A J_n(\lambda r) + B Y_n(\lambda r)$$

solutions to Bessels equ.





$B = 0$

Useless for our domain.

Annular Drum ?!?

Bessel functions of 2nd kind.

- As the θ mode increases, we move through Bessel functions of 1st kind.

  $n=0 \Rightarrow$ 1st kind J_0

  $n=1 \Rightarrow$ 1st kind J_1

  $\vdots$

- The radial mode is determined by roots of $J_n(\lambda r)$

  Roots are <u>approximately</u> spaced by 3.

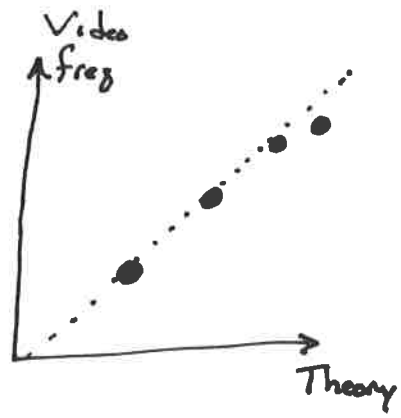
- Frequency is $\sqrt{\lambda^2 c^2} = \lambda c$

  <u>Not</u> an integer multiple of fundamental freq.

  "Drums sound different from strings"

Plot
Low to High Freq (Farlow 236)

Theory	Video	mode n,m
2.40	82.2	0, 1
3.83	158.0	1, 1
5.13	217.0	2, 1
5.52	227.0	2, 0



$$u_{tt} = c^2 \nabla^2 u$$

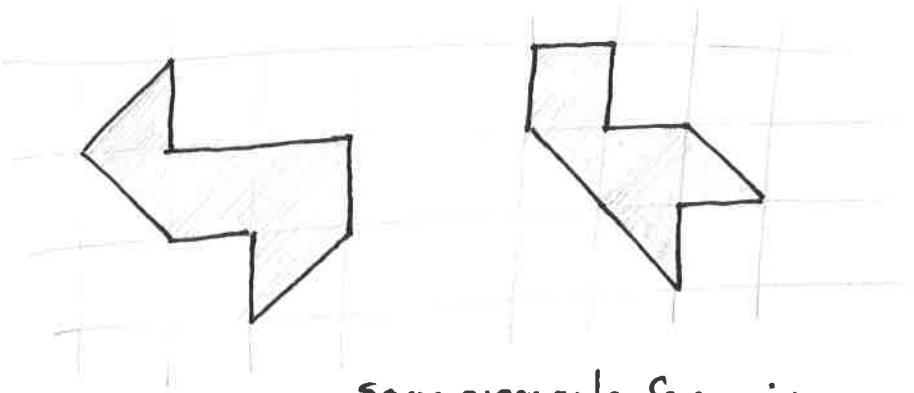
$$u(R) = 0$$

Can you hear the shape of a drum?

translation: Is the modal frequency spectrum <u>unique</u>?

<u>No</u>

Posed in 1966. Proved in 2D in 1992.



Same eigenmode frequencies

See wikipedia for more info and references.
"Hearing the shape of a drum"

Unique spectrum does <u>not</u> mean no information