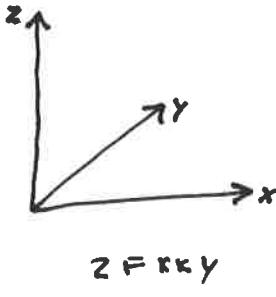


Laplacian in Various Coordinate frames.

$$\nabla^2 U$$

Cartesian (3D)



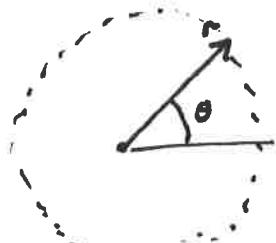
$$\nabla^2 U = \frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + \frac{d^2 U}{dz^2} = 0$$

Finite difference (2D)



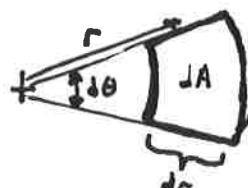
$$N + S + E + W - 4C$$

Polar (2D)



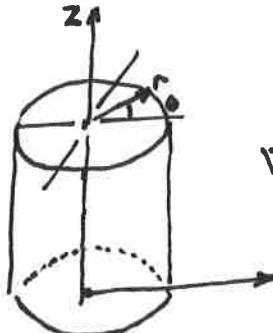
$$\nabla^2 U = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta}$$

Look at slice



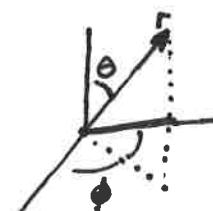
Cylindrical (3D)

Just add z to polar



$$\nabla^2 U = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + U_{zz}$$

Spherical (3D)



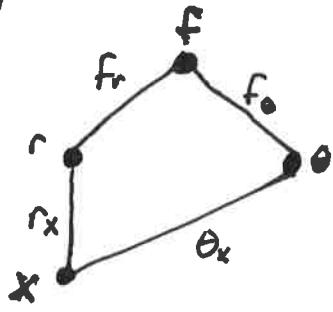
$$\begin{aligned} \nabla^2 U = & U_{rr} + \frac{2}{r} U_r + \frac{1}{r^2} U_{\theta\theta} \\ & + \frac{\cot\theta}{r^2} U_\phi + \frac{1}{r^2 \sin^2\phi} U_{\phi\phi} \end{aligned}$$

Visual Chain rule

What is $\frac{d}{dx}(f(g(x)))$? $\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}$

What is $\frac{d}{dx}(f(r, \theta))$? $\frac{df}{dx} = \frac{\partial f}{\partial r} \frac{dr}{dx} + \frac{\partial f}{\partial \theta} \frac{d\theta}{dx}$

Farlow shows an interesting visual approach to this Example.



f is a function of r, θ

x is a function of r, θ

The lines are the partial derivatives

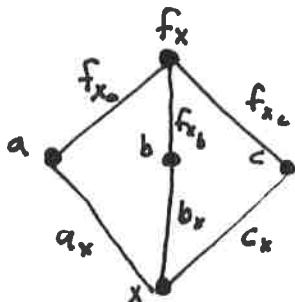
$$f_r = \frac{\partial f}{\partial r}$$

The derivative equals the sum of all paths!

$$\frac{df}{dx} = (\text{left path from } x \text{ to } f) + (\text{right path from } x \text{ to } f)$$

$$\boxed{\frac{df}{dx} = r_x \cdot f_r + \theta_x f_\theta}$$

More Complicated example (2nd deriv)



f is a function of a, b, c
 f_x is a function of a, b, c
 x is a function of a, b, c

$$\frac{df_x}{dx} = \frac{d^2 f}{dx^2} = f_{x_a} a_x + f_{x_b} b_x + f_{x_c} c_x$$

Expand

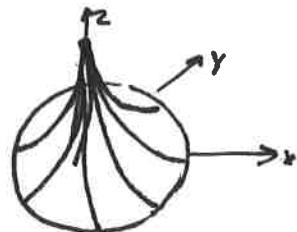
Boundary Value Problems.

The solution depends on

- Governing Equation
- Boundary Conditions

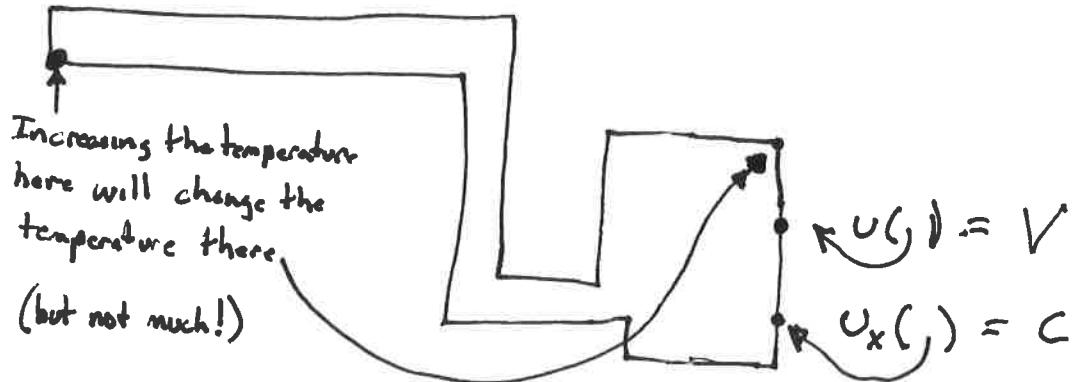
For a given Gov Egu, the solution only depends on the BCs.

Each point of the BCs contribute to the solution everywhere.



(this will lead to a powerful)
solution technique

$$\nabla^2 u = 0 \quad \text{Heat Diffusion}$$



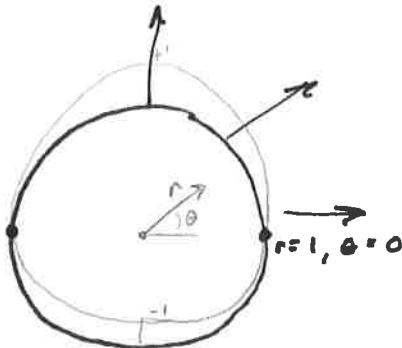
$$u(t) + u_x(t) = d$$

L32 P1.

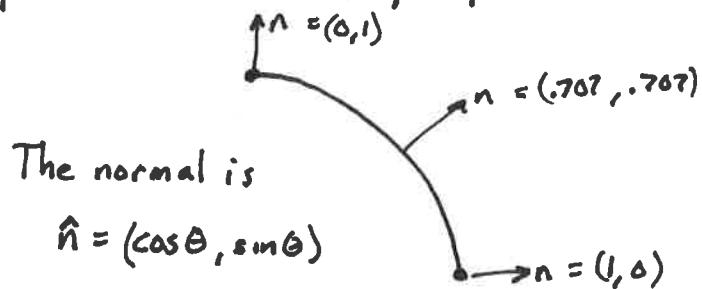
Can you find the solution to

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$u(1, \theta) = \sin \theta \quad 0 \leq \theta \leq 2\pi$$



- Wow, this appears difficult.
- Approach from a flux perspective on the boundary ($r=1$)



The y normal is exactly our BC!

This indicates that the flux $\cdot n_y$, ~~only depends~~ is constant.

- We have a Cartesian frame problem! (At least, the solution is in terms of x and y)
- Pick a function $u = f(y) = f(r \sin \theta)$

We should expect a linear function of y (diffusion).

$$u = Ay = Ar \sin \theta \Rightarrow u_r = Ar \cos \theta \quad u_{rr} = -Ar \sin \theta$$

Apply to Gov Egu $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \Rightarrow 0 + \frac{A}{r} \sin \theta + \frac{A}{r^2} (-r \sin \theta) = 0$

$$u = Ay \Rightarrow BC \stackrel{A=1}{\Rightarrow} \boxed{u = y} \boxed{= r \sin \theta}$$

L 32 P2

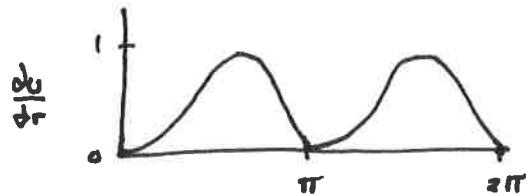
Does the following have a solution? (aka. Can a solution exist?)

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$\frac{\partial u(r=1)}{\partial r} = \sin^2 \theta$$



- The BC is visually...



This is always positive. So by inspection!

No Solution exists

- Longer explanation.

Influx of u must be matched by outflux or the system is not steady state.

$\frac{\partial u}{\partial r} = \sin^2 \theta$ is always positive, so the system is not $u_r = 0$.

L 32 P 4

Construct a BVP from

$$U_{tt} = U_{xx} - U_t + U \quad 0 < x < 1 \quad 0 < t < \infty$$

$$U(0, t) = 0$$

$$U(1, t) = 0$$

$$U(x, 0) = \sin(3\pi x)$$

$$U_t(x, 0) = 0$$



- BVPs are independent of time

$$U_t = 0, \quad U_{tt} = 0, \quad U(0, t) = U(0) = 0, \quad U(1, t) = U(1) = 0$$

- Remaining terms.

$$U_{xx} + U = 0$$

$$U(0) = 0$$

$$U(1) = 0$$



~~$$U(x) = \sin(3\pi x)$$~~

- Does a solution exist?

with BCs = 0, it is a safe bet that a solution exists.

- Is the solution unique?

No. Any $U = A \sin x$ will work.

What are we missing? $U(x, 0) = \sin(3\pi x)$ doesn't work

Separation of Variables (polar Laplacian Example)

- Domain



$$\nabla^2 u = 0 = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$u = R(r) \Theta(\theta)$$

The solution is a function of r and a function of θ

- Sep of Vars Subst' into Gov Egu

$$R_{rr} \Theta + \frac{1}{r} R_r \Theta + \frac{1}{r^2} R \Theta_{\theta\theta} = 0$$

- Pull R terms and Θ terms apart

$$\Theta \left(R_{rr} + \frac{1}{r} R_r \right) = - \frac{1}{r^2} R \Theta_{\theta\theta}$$

Divide by $\frac{R}{r^2}$ and Θ

$$\frac{r^2 (R_{rr} + \frac{1}{r} R_r)}{R} = - \frac{\Theta_{\theta\theta}}{\Theta}$$

- Again, assume that since R terms and Θ terms are independent, there is a constant that both are equal to.

$$\frac{r^2 R_{rr} + r R_r}{R} = - \frac{\Theta_{\theta\theta}}{\Theta} = + \lambda^2 \quad \text{notice that we are using } + \lambda^2 \text{ not } - \lambda^2$$

- Separated into

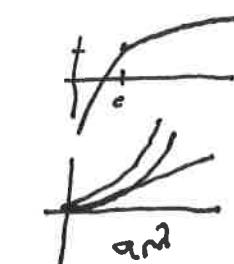
$$\underbrace{r^2 R_{rr} + r R_r}_{R} \mp \lambda^2 R = 0 \quad \text{and} \quad \underbrace{\Theta_{\theta\theta} + \lambda^2 \Theta}_{\Theta} = 0$$

New equation for us.

"Euler's Equation"

$$R(r) = a + b \ln(r) \quad \text{when } \lambda = 0$$

$$R(r) = ar^\lambda + br^{-\lambda} \quad \text{when } \lambda > 0$$



We have seen this before.

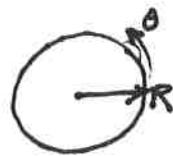
$$\Theta(\theta) = A \sin \lambda \theta + B \cos \lambda \theta$$

- What about BCs for polar Laplacian?

- Explicit BC.

$$u(R, \theta) = f(\theta)$$

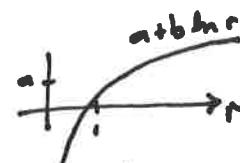
Domain



- Implicit BC. ($r=0$)

Look at solution forms.

1) $a + b \ln(r)$



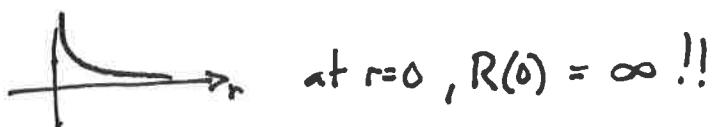
at $r=0, R(0) = -\infty !!$

2) ar^λ



well behaved in domain

3) $br^{-\lambda}$



at $r=0, R(0) = \infty !!$

For the polar Laplacian domain $0 < r < R$ only,

$$R(r) = ar^\lambda$$

- Total Solution

$$u = R(r)\Theta(\theta) = r^\lambda (a \sin \lambda \theta + b \cos \lambda \theta)$$

- Eigenvalues? $\Theta(\theta)$ must be periodic.

$$\lambda = 0, 1, 2, 3, \dots$$

