

Laplace Equation for a Dirichlet Annulus

The general solution (from Sep of Vars) is

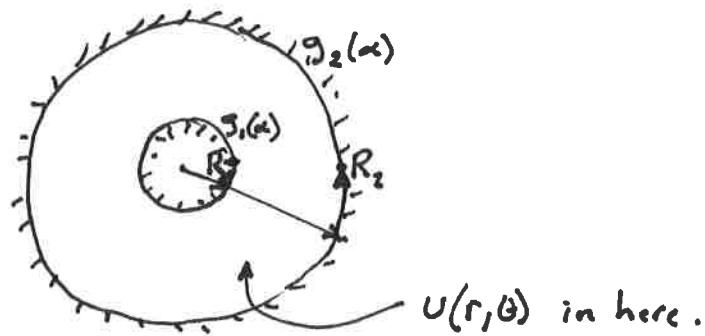
$$U(r, \theta) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta)$$

Applied to the BCs gives

$$\begin{cases} a_0 + b_0 \ln R_1 = \frac{1}{2\pi} \int_0^{2\pi} g_1(\alpha) d\alpha \\ a_0 + b_0 \ln R_2 = \frac{1}{2\pi} \int_0^{2\pi} g_2(\alpha) d\alpha \end{cases} \Rightarrow a_0 \text{ and } b_0$$

$$\begin{cases} a_n R_1^n + b_n R_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} g_1(\alpha) \cos(n\alpha) d\alpha \\ a_n R_2^n + b_n R_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} g_2(\alpha) \cos(n\alpha) d\alpha \end{cases} \Rightarrow a_n \text{ and } b_n$$

$$\begin{cases} c_n R_1^n + d_n R_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} g_1(\alpha) \sin(n\alpha) d\alpha \\ c_n R_2^n + d_n R_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} g_2(\alpha) \sin(n\alpha) d\alpha \end{cases} \Rightarrow c_n \text{ and } d_n$$



How were the coefficient integrals obtained?

$$1) \quad a_0 + b_0 \ln R_1 + \sum (a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta) = g_1(\theta)$$

Integrate over $0 \rightarrow 2\pi$ premultiply by $\sin(\theta)$ and $\cos(\theta)$

$$\int_0^{2\pi} \cos(\theta) (a_0 + b_0 \ln R_1 + \sum (a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta)) d\theta = \int_0^{2\pi} \cos(\theta) g_1(\theta) d\theta$$

only $(a_0 + b_0 \ln R_1) \theta \Big|_0^{2\pi} = \int_0^{2\pi} \cos(\theta) g_1(\theta) d\theta$

$a_0 + b_0 \ln R_1 = \frac{1}{2\pi} \int_0^{2\pi} g_1(s) ds$

Same for R_2

$$2) \quad \text{Integrate } \int_0^{2\pi} \cos(n\theta) u(r, \theta) d\theta = \int_0^{2\pi} \cos(n\alpha) g(\alpha) d\alpha$$

$$\int_0^{2\pi} \cos(n\alpha) (a_0 + b_0 \ln R_1 + \sum (a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta)) d\alpha$$

constant w/ α orthogonal, Nonzero when $n = \alpha$ orthogonal ~~always~~ always!

$$\int_0^{2\pi} \cos(n\alpha) \cos(n\theta) d\alpha (a_n R_1^n + b_n R_1^{-n}) = \int_0^{2\pi} \cos(n\alpha) g(\alpha) d\alpha$$

$\approx = 2 \int f(x) dx$

$a_n R_1^n + b_n R_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cos(n\alpha) g(\alpha) d\alpha$

Same for R_2 and sine terms.

Quick 2×2 Matrix Inverse

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and inverse exists } \det M \neq 0$$

$$\det M = ad - bc$$

$$M^{-1} = \frac{\text{switch diagonals, invert sign of off diagonals}}{\text{Divide by } \det(M)}$$

$$\boxed{M^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}}$$

Test:

$$MM^{-1} \stackrel{?}{=} [I]$$

$$\frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{\det M} = \frac{\begin{bmatrix} ad - bc & 0 \\ cd - ca & 0 \end{bmatrix} + \cancel{\begin{bmatrix} -ab + ab & 0 \\ -cb + ad & 0 \end{bmatrix}}}{ad - bc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

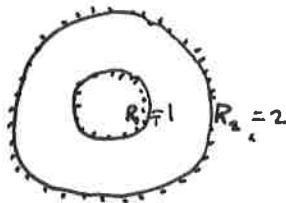
This is super useful. You should memorize it.

L34 P1 Dirichlet Annulus

$$\nabla^2 u = 0 \quad 1 < r < 2$$

$$u(1, \theta) = \cos \theta \quad R_1 = 1$$

$$u(2, \theta) = \sin \theta \quad R_2 = 2$$



First find the coefficients.

$$a_0 + b_0 \ln R_1 = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta \, d\theta = 0 \quad \Rightarrow \quad a_0 = b_0 = 0 \quad \text{By inspection}$$

$$a_0 + b_0 \ln R_2 = \dots = 0$$

$$a_n R_1^n + b_n R_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cos \theta \cos(n\theta) \, d\theta = \frac{\pi}{\pi} = 1 \quad \text{when } n=1$$

$$a_n R_2^n + b_n R_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cancel{\sin \theta} \cos(n\theta) \, d\theta = \frac{\pi}{\pi} = 0 \quad \cancel{\text{when } n \neq 1}$$

$$\begin{bmatrix} R_1 & R_1^{-1} \\ R_2 & R_2^{-1} \end{bmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \frac{\begin{bmatrix} R_2^{-1} - R_1^{-1} \\ -R_2 + R_1 \end{bmatrix}}{R_2^{-1} R_1 - R_1^{-1} R_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{R_2^{-1} - R_1^{-1}}{-R_2 + R_1} \begin{pmatrix} R_2^{-1} \\ -R_2 \end{pmatrix}$$

$$c_n R_1^n + d_n R_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} \cancel{\cos \theta} \sin(n\theta) \, d\theta = 0$$

$$c_n R_2^n + d_n R_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} \sin(\theta) \sin(n\theta) \, d\theta = \frac{\pi}{\pi} = 1 \quad \text{when } n=1$$

$$\begin{bmatrix} R_1 & R_1^{-1} \\ R_2 & R_2^{-1} \end{bmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \frac{\begin{bmatrix} R_2^{-1} - R_1^{-1} \\ -R_2 + R_1 \end{bmatrix}}{R_1 R_2^{-1} - R_2 R_1^{-1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} -R_1^{-1} \\ R_1 \end{pmatrix}}{R_1 R_2^{-1} - R_2 R_1^{-1}}$$

Substitute this into general solution.

$$u(r, \theta) = \frac{R_2^{-1} r + -R_2 r^{-1}}{R_2^{-1} R_1 - R_1^{-1} R_2} \cos \theta + \frac{-R_1^{-1} r + R_1 r^{-1}}{R_1 R_2^{-1} - R_2 R_1^{-1}} \sin \theta$$

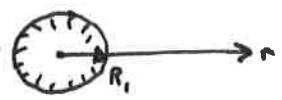
Laplace Equation for Exterior Dirichlet problem

Only keep solutions bounded as $r \rightarrow \infty$

$$U(r, \theta) = \sum_{n=0}^{\infty} r^{-n} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) d\theta \quad b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta$$



That's all.

Example:

Electrostatics.

Spherical Coordinates

$$\nabla^2 U = 0$$

Separation of Variables

$$U = R(r) \Phi(\phi) \Theta(\theta)$$

lets ignore Θ for now

$$U = R(r) \Phi(\phi)$$

- Subst into Gov Eqns.

$$\nabla^2 U = (r^2 U_r)_r + \frac{1}{\sin \phi} (\sin \phi U_\phi)_\phi = 0$$

Step by step (messy)

$$(r^2 R_r \Phi)_r + \frac{1}{\sin \phi} (\sin \phi R \Phi_\phi)_\phi = 0$$

$$\cancel{2r R_r \Phi + r^2 R_{rr} \Phi} + \frac{1}{\sin \phi} (\cos \phi R \Phi_\phi + \sin \phi R \Phi_{\phi\phi}) = 0 \quad \xrightarrow{\text{Bad idea. Leave } \frac{1}{\sin \phi} \text{ alone}}$$

$$2r R_r \Phi + r^2 R_{rr} \Phi + \frac{1}{\sin \phi} (\sin \phi R \Phi_\phi) = 0$$

mult by $\sin \phi$ and rearrange

$$(\sin \phi \Phi_\phi)_\phi R + (2r R_r + r^2 R_{rr}) \Phi \sin \phi = 0$$

Move terms

$$\frac{(\sin \phi \Phi_\phi)_\phi R'}{\phi \sin \phi} = - \frac{(2r R_r + r^2 R_{rr})}{R} = -n(n+1)$$

Separated

$$(\sin \phi \Phi_\phi)_\phi + n(n+1) \sin \phi \Phi_\phi = 0$$

$$r^2 R_{rr} + 2r R_r - n(n+1) R = 0$$

Legendre Equation

Euler's Equation

Legendre polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$\cos\phi = x$$

$$P_0(\cos\phi) = 1$$

$$P_1(\cos\phi) = \cos\phi$$

Spherical Harmonics

- Rewrite $\nabla^2 U$ as $(r^2 U_r)_r + \frac{1}{\sin\phi} (\sin\phi U_\phi)_\phi + \frac{1}{\sin^2\phi} U_{\phi\phi} = 0$

with BC $U(R, \theta, \phi) = g(\phi)$

- Decompose surface $g(\phi)$ in terms of Legendre polynomials

$$g(\phi) = a_n P_n(\cos\phi)$$

where $P_0(\cos\phi) = 1$

$$P_1(\cos\phi) = \cos\phi$$

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1)$$

$$P_3(\cos\phi) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi)$$

$$P_4(\cos\phi) = \frac{1}{8}(35\cos^4\phi - 30\cos^2\phi + 3)$$

:

See mathworld.wolfram.com for more.

- Alternatively, perform an integral to determine a_n

$$a_m = \frac{2m+1}{2} \int_0^\pi g(\phi) P_m(\cos\phi) \sin\phi d\phi$$

- Interior Solution

$$U(r, \phi) = a_n r^n P_n(\cos\phi)$$

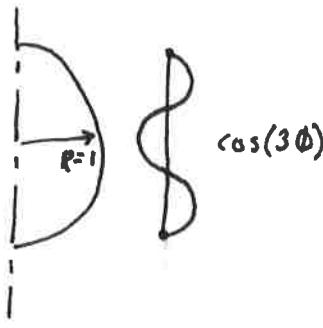
- Exterior Solution

$$U(r, \phi) = \frac{a_n}{r^{n+1}} P_n(\cos\phi)$$

L35 P4

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$u(1, \phi) = \cos(3\phi)$$



General solution form

$$u(r, \theta, \phi) = \sum a_n r^n P_n(\cos \phi)$$

Apply Integral with premultiplied function

$$\int_0^\pi P_n(\cos \phi) u(r, \theta, \phi) d\phi \underset{\sum a_n r^n P_n \cos \phi}{=} \int_0^\pi \underbrace{\sum a_n r^n P_n \cos \phi}_{\text{orthogonal except when } \phi = \alpha} P_n \cos \phi d\phi$$

$$a_m = \frac{2m+1}{2} \int_0^\pi \cos(3\phi) P_m(\cos \phi) \sin(\phi) d\phi$$

$\uparrow = 4\cos^3 \phi - 3\cos \phi$

$$P_0(\cos \phi) = 1$$

$$P_1(\cos \phi) = \cos \phi$$

$$P_2(\cos \phi) = \frac{1}{2}(3\cos^2 \phi - 1)$$

$$P_3(\cos \phi) = \frac{1}{2}(5\cos^3 \phi - 3\cos \phi)$$

$$a_0 = \frac{1}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)(1) \sin \phi d\phi$$

even even odd

$$= 0$$

$$a_1 = \frac{3}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)(\cos \phi)(\sin \phi) d\phi = -\frac{3}{5}$$

$$a_2 = \frac{5}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)\left(\frac{1}{2}\right)(3\cos^2 \phi - 1) \sin \phi d\phi = 0$$

$$a_3 = \frac{7}{2}\left(\frac{16}{35}\right) = \frac{8}{5}$$

$$a_4 = 0$$

$$a_5 = 0$$

Quick Lookup Method to find a_n for Legendre Poly'

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}(P_0(x) + 2P_2(x))$$

$$x^3 = \frac{1}{5}(3P_1(x) + 2P_3(x))$$

$$x^4 = \frac{1}{35}(7P_0(x) + 20P_2(x) + 8P_4(x))$$

$$x^5 = \frac{1}{63}(27P_1(x) + 28P_3(x) + 8P_5(x))$$

Consider these the "inverse" of the P_n definitions.

Find more at:

[mathworld.wolfram.com/
Legendre Polynomial.html](http://mathworld.wolfram.com/LegendrePolynomial.html)

Example:

$$g(\phi) = 4\underbrace{\cos^3(\phi)}_{x^3} - 3\underbrace{\cos(\phi)}_x \quad \text{with } x = \cos(\phi)$$

$$g(\phi) = 4\left(\frac{1}{5}(3P_1 + 2P_3)\right) - 3(P_1) = \underbrace{\frac{12}{5}P_1 + \frac{8}{5}P_3}_{\frac{12}{5} - \frac{15}{5} = -\frac{3}{5}} - 3P_1 = -\frac{3}{5}P_1 + \frac{8}{5}P_3$$

$$\boxed{g(\phi) = 4\cos^3(\phi) - 3\cos(\phi) = -\frac{3}{5}P_1(\cos(\phi)) + \frac{8}{5}P_3(\cos(\phi))}$$

Example

$$g(\phi) = \cos(5\phi) = 16\cos^5\phi - 20\cos^3\phi + 5\cos\phi$$

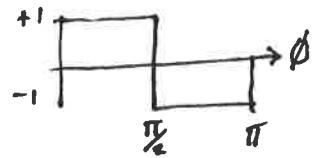
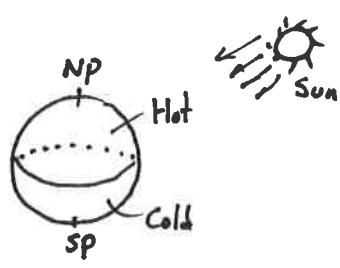
$$= 16\left(\frac{1}{63}\right)(27P_1 + 28P_3 + 8P_5) - 20\left(\frac{1}{5}\right)(3P_1 + 2P_3) + 5P_1$$

$$\boxed{= -\frac{1}{7}P_1(\cos(\phi)) - \frac{8}{9}P_3(\cos(\phi)) + \frac{128}{63}P_5(\cos(\phi))}$$

L35 P5 Solve Hot/Cold Earth "August"

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$u(l, \phi) = \begin{cases} 1 & 0 \leq \phi \leq \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < \phi \leq \pi \end{cases}$$



- We need to represent the surface BC in terms of the Legendre Polynomial (of $\cos\phi$)
 - A series expansion of $\cos(n\phi)$ converted to P_n would be messy.
 - Direct integral is a better bet

$$a_m = \frac{2m+1}{2} \int_0^{\pi/2} P_m(\cos\phi) \sin\phi d\phi + \frac{2m+1}{2} \int_{\pi/2}^{\pi} -P_m(\cos\phi) \sin\phi d\phi$$

- We need a way to obtain P_n as n increases.

$$P_{n+1} = \frac{2n+1}{n+1}(x)P_n(x) - \frac{n}{n+1}P_{n-1}$$

or

$$P_m = \frac{2m-1}{m} \cos\phi P_{m-1}(\cos\phi) - \frac{m-1}{m} P_{m-2}(\cos\phi)$$

The recurrence equation for P is
 (The Farlow book presents the Rodrigues formula. I don't prefer using it! Recurrence is easier to code.) p284

Example

$$P_2 = \frac{3}{2} \cos\phi \cos\phi - \frac{1}{2} \quad \checkmark$$

- Best solved with a computer except for very simple BCs