

Lesson 33

Theory Laplacian PDE

Interior Dirichlet, Polar Coordinates

$$\nabla^2 U = 0 \quad 0 < r < 1$$

$$U(1, \theta) = f(\theta) \quad 0 < \theta < 2\pi$$

Fig 33.1 in Farlow

Sep of Vars

$$U(r, \theta) = R(r) \Theta(\theta) \Rightarrow R_{rr} \Theta + \frac{1}{r} R_r \Theta + \frac{1}{r^2} \Theta_{\theta\theta} = 0$$

or

$$\frac{r^2 R_{rr}}{R} + \frac{r R_r}{R} = - \frac{\Theta_{\theta\theta}}{\Theta} = \lambda^2$$

or

$$r^2 R_{rr} + r R_r - \lambda^2 R = 0$$

and

$$\Theta_{\theta\theta} + \lambda^2 \Theta = 0$$



Harmonic solution (period 2π)

$$\Theta = A \sin \lambda \theta + B \cos \lambda \theta$$

$$\lambda = 0, 1, 2, \dots n$$

Euler's Equation / ODE
see Farlow p 272

For an interior dirichlet case

$$R_n(r) = a_n r^n$$

$$U = R(r) \Theta(\theta)$$

$$U(r, \theta) = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Poisson Integral Formula

- Solve on a circle

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad 0 < r < R$$

$$U(R, \theta) = g(\theta) \quad 0 \leq \theta < 2\pi$$

Separation of Variables solution is

$$U(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

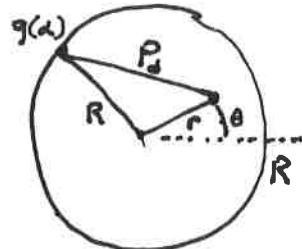
$$\boxed{\nabla^2 U = 0}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) d\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta$$

Now substitute and add magic.

Fig 33.2



$$U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2}}_{\text{Weighting}} \underbrace{g(\alpha) d\alpha}_{BC}$$

- So the solution is a weighted average of the Boundary Conditions.

- $U(r=0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha = \text{Average Value of } g(\alpha)$

- $U(r=R, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{0}_{2R^2 - 2R^2 \cos(\theta - \alpha)} g(\alpha) d\alpha \stackrel{?}{=} 0 \stackrel{\text{No}}{=}$

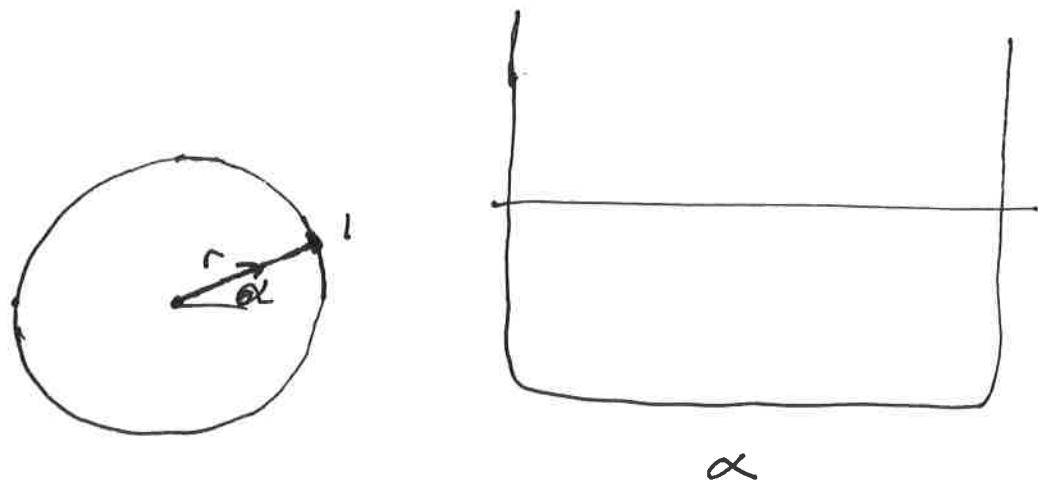
Only zero when $\theta \neq \alpha$, otherwise must be ~~zero~~ $U(r=R, \theta=\alpha) = g(\alpha)$

But isn't $\frac{0}{0}$ undefined? Don't use Poisson on/near boundary!

To be clear, the Poisson Integral Formula does the exact same thing as a series solution (even being derived from a series solution.). The solution is identical. However, near the boundary, the Poisson Formula breaks down. Don't use it near a boundary (say $r > 0.95 R$).

Demo.

PoissonIntegral.py



matplotlib.org/gallery

L33 P2

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad 0 < r < 1$$

a) $U(1, \theta) = 1 + \sin \theta + \frac{1}{2} \cos \theta$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \cos(\theta) \text{ term} & \sin(\theta) \text{ term} & \cos(\theta) \text{ term.} \end{matrix}$

$$\begin{matrix} a_0 = 1 & b_1 = 1 \\ a_1 = \frac{1}{2} & \text{all others} = 0 \end{matrix}$$

$$U(r, \theta) = 1 + r \left(\frac{1}{2} \cos \theta + \sin \theta \right)$$

$$\text{from } U(r, \theta) = \sum r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Can you imagine the integral for the Poisson Integral formula? Nasty!

b) $U(1, \theta) = 2$

By inspection

$$U(r, \theta) = 2$$

Boring....

c) $U(1, \theta) = \sin(\theta)$ By inspection! sine in BC and series is identical

$$U(r, \theta) = r \sin \theta$$

d) $U(1, \theta) = \sin(3\theta)$

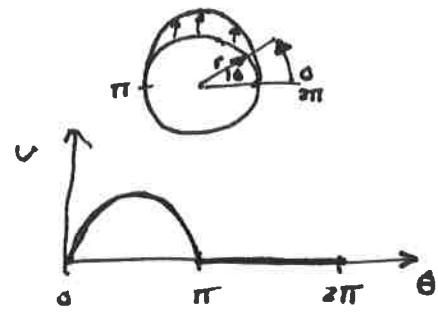
$$U(r, \theta) = r^3 \sin(3\theta)$$

Now this is interesting!

L33 p5

$$\nabla^2 U = 0 \quad 0 < r < 1$$

$$U(r, \theta) = \begin{cases} \sin \theta & 0 \leq \theta < \pi \\ 0 & \pi \leq \theta < 2\pi \end{cases}$$



- Find a sine expansion of the BCs.

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \sin(\theta) \sin(n\theta) d\theta + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \sin(n\theta) d\theta$$

$$b_1 = \frac{1}{2}, \text{ all others } = 0 \quad (\text{why is this different from previous sine expansions?})$$

- Find a cosine expansion

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta = \frac{1}{\pi}$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} \sin(\theta) \cos(n\theta) d\theta = 0$$

$$a_2 = -\frac{2}{3\pi}, \quad a_3 = 0, \quad a_4 = -\frac{2}{15\pi}$$

$$a_{\text{even}} = \frac{-2}{(n^2-1)\pi}$$

- Solution

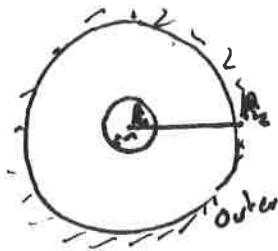
$$U(r, \theta) = \frac{1}{\pi} + r \left(\frac{y_2}{2} \sin \theta \right) + r^2 \left(-\frac{2}{3\pi} \cos 2\theta \right) + r^4 \left(-\frac{2}{15\pi} \cos 4\theta \right) + \dots$$

Demo

L33p5-derive.py

L33p5.py

Lesson 34. Laplace's Equation (Dirichlet BCs) in an annulus



Inner radius R_i
Outer radius R_o

Remember that the Sep of Vars Eqn was

$$r^2 R'' + rR' - \lambda^2 R = 0$$

And the solutions were.

- $\lambda = 0$

$$R(r) = a + b \ln(r)$$

- $\lambda > 0$

$$R(r) = ar^\lambda + br^{-\lambda}$$

- We threw out all but the ar^λ solution and constant a solution because the BCs gave infinite values.
- Now, we keep these solutions.

$$U(r, \theta) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[(a_n r^n + b_n \bar{r}^n) \cos n\theta + (c_n r^n + d_n \bar{r}^{-n}) \sin n\theta \right]$$