

Green's Function (aka. Impulse Response Function)

- Context: Remember the definition of an antiderivative

$$f = \int g(x)dx \text{ means we are looking for } \frac{df}{dx} = g$$

- Greens ~~function~~ function is similar

$$L G = \delta \quad \text{we are looking for a function } G \text{ that the PDE turns into } \delta \text{ (delta operator)}$$

- L is a linear operator

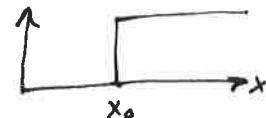
For diffusion $U_t = \alpha^2 U_{xx}$ then $L = \partial_t - \alpha^2 \partial_{xx}$

- G is the Green's function!

- $\delta(x - x_0)$ is a delta operator/function defined as

$$\int_{-\infty}^{\infty} \delta(x - x_0) = H(x - x_0) \text{ and } \delta(x - x_0) = 0 \text{ except at } x = x_0$$


Also $H(x - x_0)$ is the Heaviside function

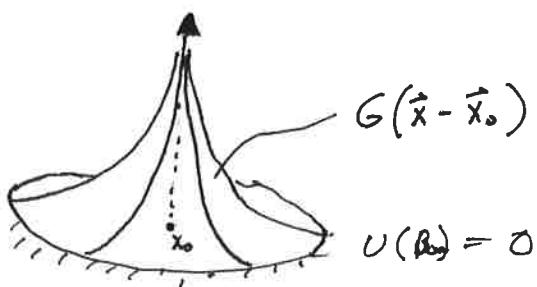


Example:

$$\text{so } \frac{d}{dx} H(x-2) = \delta(x-2)$$

- Alternative View

Green's function is the solution to a PDE (or ODE) with an impulse input.



Greens function is specific to a given L , and BCs.

G is tabulated for many popular equations and BCs. (wikipedia)

L	G
$\partial_t + \gamma$	$H(t) e^{-\gamma t}$
$\partial_{tt} + 2\gamma \partial_t + \omega_0^2$	$H(t) e^{-\gamma t} \frac{1}{\omega} \sin(\omega t)$ for $\omega = \sqrt{\omega_0^2 - \gamma^2}$
$\nabla^2 V_{2D}$	$\frac{1}{2\pi} \ln r$
$\nabla^2 V_{3D}$	$-\frac{1}{4\pi r}$
$\nabla^2 + k^2$	$-\frac{e^{-ikr}}{4\pi r}$
$\frac{1}{c^2} \partial_{tt} - \nabla^2$	$\frac{\delta(t - \frac{r}{c})}{4\pi r}$
$\partial_t - \alpha^2 \nabla^2$	$H(t) \left(\frac{1}{4\pi \alpha^2} \right)^{1/2} e^{-\frac{r^2}{4\alpha^2} t}$

Example.

Verify G for ~~the~~ simple ODE

$$L = \partial_t + \gamma \rightarrow H(t)e^{-\gamma t}$$

$$L G \stackrel{?}{=} \delta(t)$$

Substitute

$$\partial_t(H(t)e^{-\gamma t}) + \gamma H(t)e^{-\gamma t}$$

Chain rule

$$\partial_t(H(t)) e^{-\gamma t} + H(t)(-\gamma) e^{-\gamma t} + \gamma H(t) e^{-\gamma t}$$

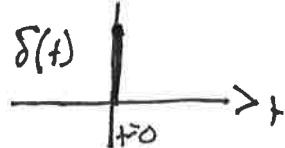
Identity for Heavyside function

$$\cancel{\delta(t)} e^{-\gamma t}$$

Also impulse is only nonzero at $t=0$, so $\delta(t) = 1$

$$\boxed{\delta(t) = L G}$$

Verified



Verify $\mathcal{L}G = \delta$ for 2D Laplacian

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$G = \frac{1}{2\pi} \ln(r)$$

subs.

$$\mathcal{L}G = \frac{-1}{2\pi r^2} + \frac{1}{2\pi} \frac{1}{r^2} + 0 = 0$$

- But you say "wait, this is wrong. Where is δ ?". Good question.

$\ln(r)$ is not defined at $r=0$! We forgot to include that point $r=0$.

- How can we find the derivative when the function isn't defined?

- Complex Variables

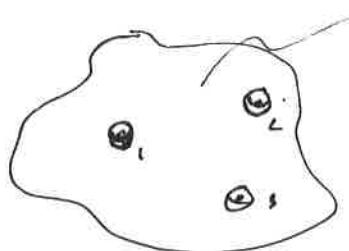
$$\ln(x+i0) \xrightarrow{\text{zero!}} \lim_{\epsilon \rightarrow 0} \ln(x+i\epsilon) = \ln|x| + i\pi H(-x) \xrightarrow{\text{branch cut!}}$$

Final result

$$\frac{d}{dx} \ln(x) = \frac{1}{x} - i\pi \delta(x)$$

$$\frac{d^2}{dx^2} \ln(x) = -\frac{1}{x^2} - i\pi \frac{d}{dx} \delta(x) \xrightarrow{\text{unit doublet}}$$

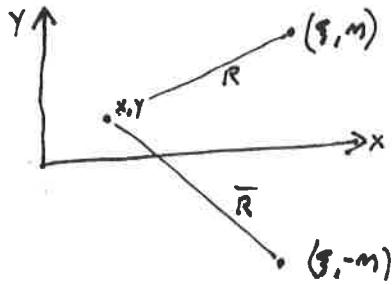
For more information, read Salsa's 3.5.1



$$= V + S_1 + S_2 + S_3$$

L36 p2

Find $G(x, y, \xi, \eta)$ for ∇^2 in the upper plane ($y > 0$)



point charge = point source

$$u(r) = \frac{q}{2\pi} \ln \frac{1}{r} = -\frac{q}{2\pi} \ln r$$

$$\text{From figure } R = \sqrt{(x-\xi)^2 + (y-\eta)^2} \quad \text{and} \quad \bar{R} = \sqrt{(x-\xi)^2 + (y+\eta)^2}$$

Total

$$U(r) = -\frac{q}{2\pi} \ln(R) + -\frac{\bar{q}}{2\pi} \ln(\bar{R})$$

$$= -\frac{1}{2} \frac{q}{2\pi} \ln \left(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \right)$$

We want $U(r)$ to be $+\infty$ and 0 on $y=0$

$$\xi = -1$$

$$\ln \left(\frac{\dots + \dots}{\dots + \dots} \right) = \ln(1) = 0$$

$$G(x, y, \xi, \eta) = -\frac{1}{2\pi} \ln \left(\frac{R}{\bar{R}} \right)$$

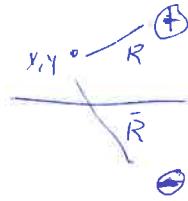
$$= \frac{1}{2\pi} \ln \left(\frac{\bar{R}}{R} \right)$$

$$= \frac{1}{2\pi} \ln \left(\frac{\sqrt{(x-\xi)^2 + (y+\eta)^2}}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \right)$$

L36 P3

$$\nabla^2 U = -k \quad \text{in upper-half plane}$$

$$G = \frac{1}{2\pi} \ln \left(\frac{R}{r} \right)$$



$$U(x, y) = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{2\pi} \ln \left(\frac{R}{r} \right) (-k) dr dy$$

Fredholm Theory for finding Green's functions of Sturm-Liouville problems.

Given a Sturm-Liouville ODE, we can find $G(x, x')$!

1) Determine the eigenfunctions and eigenvalues
 ψ λ

2) $G(x, x') = \sum_{n=0}^{\infty} \frac{\psi_n(x) \psi_n(x')}{\lambda_n}$

Example:

$$y'' + \lambda^2 y = 0 \quad \text{with } y(0) = 0 \text{ and } y(1) = 0$$

$$y'' + n^2 \pi^2 y = 0 \Rightarrow y(x) = \psi = \sin(n\pi x)$$
$$\lambda_n = n\pi$$

From above

$$G(x, x') = \sum_{n=0}^{\infty} \frac{\sin(n\pi x) \sin(n\pi x')}{n\pi}$$

See Demo.

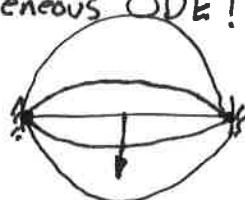
fredholm.py

Big Idea:

Given $G(x, x')$, you can solve a non-homogeneous ODE!

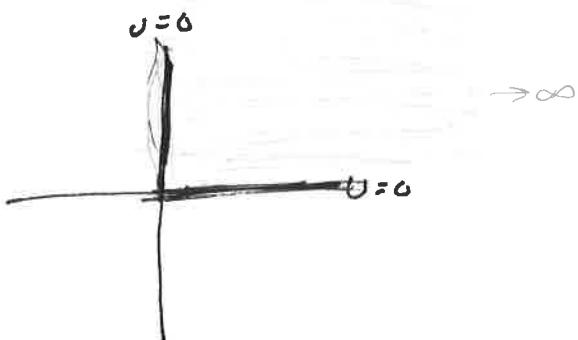
$$y'' + \lambda^2 y = f(x)$$

$$y(x) = \int_0^x G(x, x') f(x') dx'$$

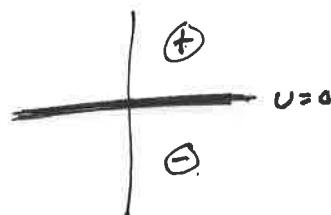


Green's Functions Redux.

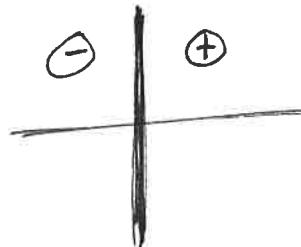
How can we solve L36P4? $G(x,y,\xi,\eta)$ on upper-right quadrant.



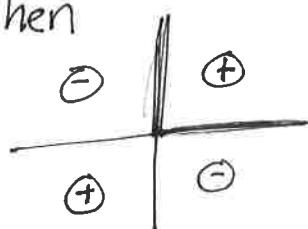
If



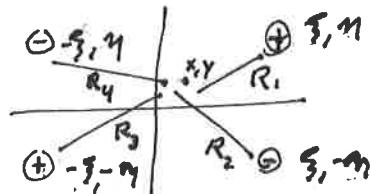
and rotated



Then



By inspection!



Check this. $\phi = \frac{1}{2\pi} \ln \frac{1}{r}$

$$\phi = \frac{1}{2\pi} \left(\ln \frac{1}{R_1} - \ln \frac{1}{R_2} + \ln \frac{1}{R_3} - \ln \frac{1}{R_4} \right)$$

$$= \frac{1}{2\pi} \ln \left(\frac{R_2 R_4}{R_1 R_3} \right) \quad \underline{\text{Expand this}}$$

$$= \frac{1}{2\pi} \ln \left(\frac{\sqrt{(x-s)^2 + (y+\eta)^2} \sqrt{(x+s)^2 + (y-\eta)^2}}{\sqrt{(x-s)^2 + (y-\eta)^2} \sqrt{(x+s)^2 + (y+\eta)^2}} \right)$$