

Lesson 35
Spherical Harmonics

Spherical Coordinates

$$\nabla^2 U = 0$$

Separation of Variables

$$U = R(r) \Phi(\phi) \Theta(\theta)$$

lets ignore Θ for now

$$U = R(r) \Phi(\phi)$$

- Subst into Gov Equ.

$$\nabla^2 U = (r^2 U_r)_r + \frac{1}{\sin \phi} (\sin \phi U_\phi)_\phi = 0$$

Step by step (messy)

$$(r^2 R_r \Phi)_r + \frac{1}{\sin \phi} (\sin \phi R \Phi_\phi)_\phi = 0$$

$$\cancel{2r R_r \Phi + r^2 R_{rr} \Phi} + \frac{1}{\sin \phi} (\cos \phi R \Phi_\phi + \sin \phi R \Phi_{\phi\phi}) = 0$$

Bad idea. Leave $\frac{1}{\sin \phi}$ alone

$$2r R_r \Phi + r^2 R_{rr} \Phi + \frac{1}{\sin \phi} (\sin \phi R \Phi_\phi) = 0$$

mult by $\sin \phi$ and rearrange

$$(\sin \phi \Phi_\phi)_\phi R + (2r R_r + r^2 R_{rr}) \Phi \sin \phi = 0$$

Move terms

$$\frac{(\sin \phi \Phi_\phi)_\phi R'}{\Phi \sin \phi} = - \frac{(2r R_r + r^2 R_{rr})}{R} = -n(n+1)$$

Separated

$$(\sin \phi \Phi_\phi)_\phi + n(n+1) \sin \phi \Phi = 0$$

$$r^2 R_{rr} + 2r R_r - n(n+1) R = 0$$

Legendre Equation

Euler's Equation

Spherical Harmonics

- Rewrite $\nabla^2 U$ as $(r^2 U_r)_r + \frac{1}{\sin\phi} (\sin\phi U_\phi)_\phi + \frac{1}{\sin^2\phi} U_{\theta\theta} = 0$

with BC $U(R, \theta, \phi) = g(\phi)$

- Decompose surface $g(\phi)$ in terms of Legendre polynomials

$$g(\phi) = a_n P_n(\cos\phi)$$

where $P_0(\cos\phi) = 1$

$$P_1(\cos\phi) = \cos\phi$$

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1)$$

$$P_3(\cos\phi) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi)$$

$$P_4(\cos\phi) = \frac{1}{8}(35\cos^4\phi - 30\cos^2\phi + 3)$$

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See mathworld.wolfram.com for more.

- Alternatively, perform an integral to determine a_n

$$a_m = \frac{2m+1}{2} \int_0^\pi g(\phi) P_m(\cos\phi) \sin\phi d\phi$$

- Interior Solution

$$U(r, \phi) = a_n r^n P_n(\cos\phi)$$

- Exterior Solution

$$U(r, \phi) = \frac{a_n}{r^{n+1}} P_n(\cos\phi)$$

Quick Lookup Method to find a_n for Legendre Poly'

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}(P_0(x) + 2P_2(x))$$

$$x^3 = \frac{1}{5}(3P_1(x) + 2P_3(x))$$

$$x^4 = \frac{1}{35}(7P_0(x) + 20P_2(x) + 8P_4(x))$$

$$x^5 = \frac{1}{63}(27P_1(x) + 28P_3(x) + 8P_5(x))$$

Consider these the "inverse" of the P_n definitions.

Find more at:
[mathworld.wolfram.com/
 Legendre Polynomial.html](http://mathworld.wolfram.com/LegendrePolynomial.html)

Example:

$$g(\phi) = 4\underbrace{\cos^3(\phi)}_{x^3} - 3\underbrace{\cos(\phi)}_x \quad \text{with } x = \cos(\phi)$$

$$g(\phi) = 4\left(\frac{1}{5}(3P_1 + 2P_3)\right) - 3(P_1) = \frac{12}{5}P_1 + \frac{8}{5}P_3 - 3P_1 = \frac{12}{5}P_1 - \frac{15}{5}P_1 + \frac{8}{5}P_3 = -\frac{3}{5}P_1 + \frac{8}{5}P_3$$

$$\boxed{g(\phi) = 4\cos^3(\phi) - 3\cos(\phi) = -\frac{3}{5}P_1(\cos(\phi)) + \frac{8}{5}P_3(\cos(\phi))}$$

Example

$$g(\phi) = \cos(5\phi) = 16\cos^5\phi - 20\cos^3\phi + 5\cos\phi$$

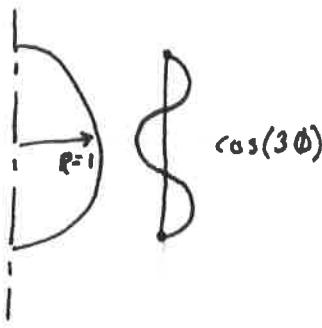
$$= 16\left(\frac{1}{63}(27P_1 + 28P_3 + 8P_5)\right) - 20\left(\frac{1}{5}(3P_1 + 2P_3)\right) + 5P_1$$

$$\boxed{= -\frac{1}{7}P_1(\cos(\phi)) - \frac{8}{9}P_3(\cos(\phi)) + \frac{128}{63}P_5(\cos(\phi))}$$

L35 P4

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$u(1, \phi) = \cos(3\phi)$$



General solution form

$$u(r, \theta, \phi) = \sum a_n r^n P_n(\cos \phi)$$

Apply Integral with premultiplied function

$$\int_0^\pi P_n(\cos \phi) u(r, \theta, \phi) d\phi \approx \int_0^\pi \sum a_n r^n P_n(\cos \phi) P_n(\cos \phi) d\phi$$

$\underbrace{\qquad\qquad\qquad}_{\text{orthogonal except when } \phi = \alpha}$

$$a_m = \frac{2m+1}{2} \int_0^\pi \cos(3\phi) P_m(\cos \phi) \sin(\phi) d\phi$$

$\uparrow = 4\cos^3 \phi - 3\cos \phi$

- $P_0(\cos \phi) = 1$
- $P_1(\cos \phi) = \cos \phi$
- $P_2(\cos \phi) = \frac{1}{2}(3\cos^2 \phi - 1)$
- $P_3(\cos \phi) = \frac{1}{2}(5\cos^3 \phi - 3\cos \phi)$

$$a_0 = \frac{1}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)(1) \sin \phi d\phi$$

even even odd

$$= 0$$

$$a_1 = \frac{3}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)(\cos \phi)(\sin \phi) d\phi = -\frac{3}{5}$$

$$a_2 = \frac{5}{2} \int_0^\pi (4\cos^3 \phi - 3\cos \phi)(\frac{1}{2})(3\cos^2 \phi - 1) \sin \phi d\phi = 0$$

$$a_3 = \frac{7}{2} \left(\frac{16}{35} \right) = \frac{8}{5}$$

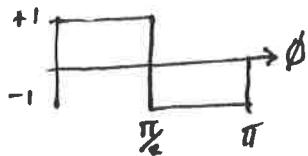
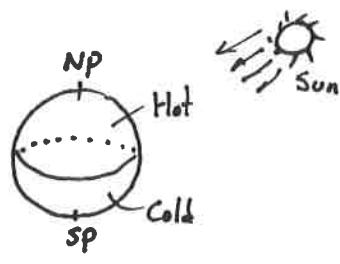
$$a_4 = 0$$

$$a_5 = 0$$

L35 P5 Solve Hot/Cold Earth "August"

$$\nabla^2 u = 0 \quad 0 < r < 1$$

$$u(l, \phi) = \begin{cases} 1 & 0 \leq \phi \leq \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < \phi \leq \pi \end{cases}$$



- We need to represent the surface BC in terms of the Legendre Polynomial (of $\cos\phi$)
 - A series expansion of $\cos(n\phi)$ converted to P_n would be messy.
 - Direct integral is a better bet

$$a_m = \frac{2m+1}{2} \int_0^{\pi/2} P_m(\cos\phi) \sin\phi d\phi + \frac{2m+1}{2} \int_{\pi/2}^{\pi} -P_m(\cos\phi) \sin\phi d\phi$$

- We need a way to obtain P_n as n increases. The recurrence equation for P is

$$P_{n+1} = \frac{2n+1}{n+1}(x)P_n(x) - \frac{n}{n+1}P_{n-1}$$

(The Farlow book presents the Rodrigues formula. I don't prefer using it! Recurrence is easier to code.) P284

or

$$P_m = \frac{2m-1}{m} \cos\phi P_{m-1}(\cos\phi) - \frac{m-1}{m} P_{m-2}(\cos\phi)$$

Example

$$P_2 = \frac{3}{2} \cos\phi \cos\phi - \frac{1}{2} \quad \checkmark$$

- Best solved with a computer except for very simple BCs