GES 554

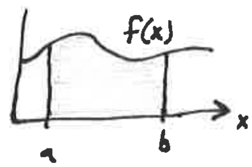
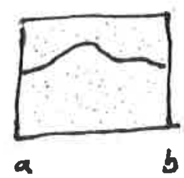
13th March 2015

Monte Carlo

A famous casino in the country of Monaco.

What can you do with 100 000 random numbers?

- Statistical Integration

$\int_a^b f(x)\,dx \approx$  $=$  $H > \max f(x)$

$$I = \frac{\text{Ratio of hits}}{H(b-a)}$$

- Cryptography (One-time-pad) (ie. shared random sequence)

Input: ATTACKATDAWNXXX

↓

Integer: 1 20 20 1 3 11 1 20 4 1 23 14 24 24 24

↓

Reversible function of random #: 3 12 24 5 6 24 1 14 6 19 23 5 8 16 2 ← random but shared

Shared SECRET!

(eg. XOR binary representation of integer and random #)

(eg. shift mod 26)

↓

Output: 4 6 18 6 9 9 2 8 10 20 20 19 6 14 26

Send this ....

Output: 4 6 18 6 9 9 2 8 10 20 20 19 6 14 26

↓

Reversible function of shared random #: 3 12 24 5 6 24 1 14 6 19 23 5 8 16 2

1 20

↓

Input: ATTACK AT DAWN XXX

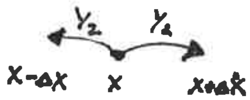
XOR Attack

Russian story

Please don't use this in any production/industry setting!

Tour du wino or Brownian Motion Simulation

Say that a particle has equal probability to move either left or right,

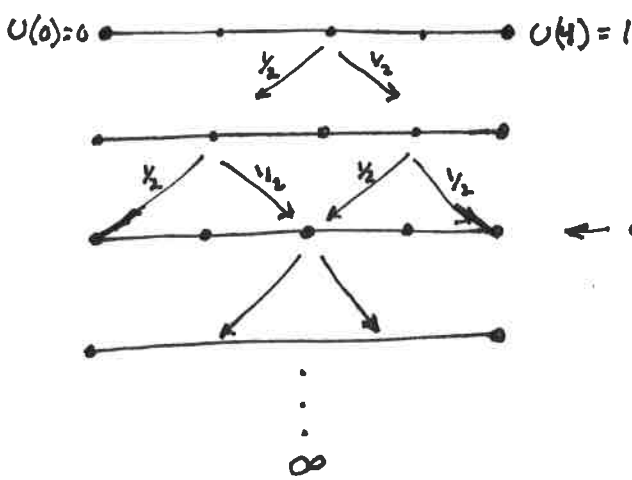


From Salsa, this is Brownian motion (e.g. particle motion) with the macro solution of diffusion. $U_t = D U_{xx}$

So, we can solve diffusion and Laplacian problems with random motion of "particles".

Example.

$\nabla^2 U = U_{xx} = 0$



← once we hit a boundary, stop.

In-Class example:

Solve $U_{xx} + 0.1\,U_x = 0$

Solution:

$$U(x) = \frac{-1}{1-e^{0.1}}\,e^{0.1x} + \frac{1}{1-e^{0.1}}$$

$U(0.5) =$

1) Start at X_0
2) Move +0.1
3) Flip Quarter
4) Move +1 if Heads or -1 if Tails
5) If you hit or go past a boundary (± 3), STOP and record the boundary.
6) Repeat N times
7) Average all N values starting at X_0

Results.

step	location	move	move
0	0	0.1	
		0.1	
		0.1	

or

	Particle #1	#2	#3
i	Total location		
0	0		
1			
2			
3			
4			

Problems with Monte Carlo

- Source of random #s. Biased?

- Solution is a set of discrete data points. Scatter.

- Error ~~variance~~ std dev decreases with $\sqrt{N}$. Many many many pts needed.

- Solution will never be identical between runs. (if random is truly random)

- Solution resembles molecular motion, so the equivalent Δt is a strong influence on the scatter of the solution.

Advantages.

- Valid for complicated PDEs

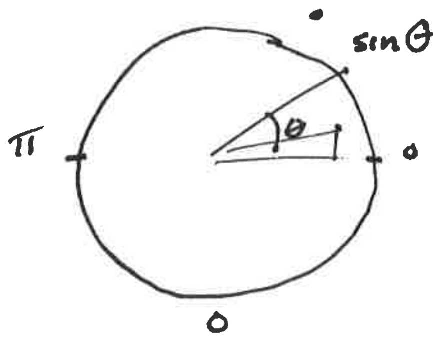
- Similar to molecular behavior. We can use a variant for high Knudsen # flows!! See "Direct Simulation Monte Carlo" DSMC.

- Output naturally has variance (scatter). We can use the output to test actual hardware or sensors. Fake hardware in the loop. (HIL)

- Extend this concept for turbulence modeling for Computational Fluid Dynamics (CFD).

How to create a Monte Carlo Solver.

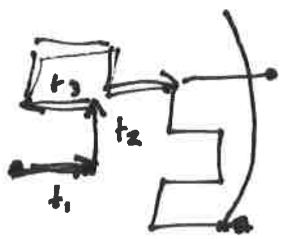
1)



$x, y \Rightarrow \theta$

$\theta = \mathrm{atan}\,\frac{y}{x}$

2)



- Move Particle ( )
  1) random # between 1, 4
  2) N, S, E, W

     N: y + delta
     S: y − delta
     E: x + delta
     W: x − delta
  3) Return x, y moved

- Move particle until hits BC.
  1) x, y = move(x, y)
  2) Check inside domain
  3) return x, y

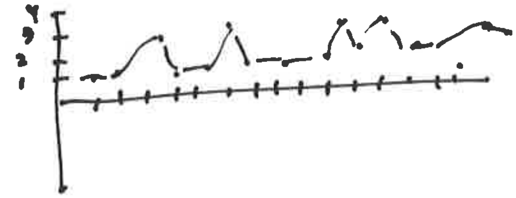
- Apply BC.
  1) $\theta = \mathrm{atan}\,\frac{y}{x}$
  2) $BC = f(\theta)$

- Multiple particles (N)
  1) loop 1 … N
     a) Apply BC (Move Particle(move))
  2) Average BC found in # 1
  3) Return MC ave

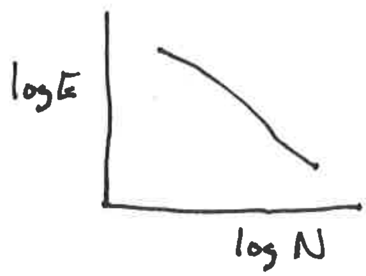
- Random # generator

  Hard to generate

  Easy to fail



- Slow convergence



if $\text{Error} \approx \frac{1}{N}$

$$\log \text{Error} \approx \log \frac{1}{N} = -\log N$$

if $\text{Error} \approx \frac{1}{\sqrt{N}}$

$$\log \text{Error} \approx \log \frac{1}{\sqrt{N}} = -\frac{1}{2} \log N$$

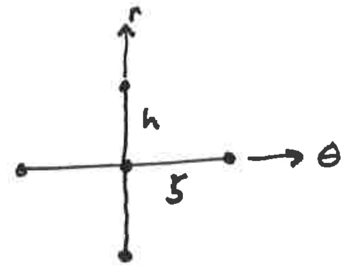
Compare analytical terms, finite difference, and Monte Carlo.
fastest med slowest

- Time to solve as $h \to 0$

- Error is a function of h and N.

Polar Coordinate Version

$$\boxed{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0}$$



$$\frac{U_N - 2U_c + U_S}{h^2} + \frac{1}{r}\,\frac{U_N - U_S}{h} + \frac{1}{r^2}\,\frac{U_W + U_E - 2U_c}{s^2} = 0$$

Solve for U_c

mult by $h^2 r^2$

$$0 = r^2(U_N + U_S) - 2U_c r^2 + rh(U_N - U_S) + \frac{h^2}{s^2}(U_W + U_E) - 2U_c\frac{h^2}{s^2}$$

$$\left(2r^2 + 2\frac{h^2}{s^2}\right)U_c = (r^2 + rh)U_N + (r^2 - rh)U_S + \frac{h^2}{s^2}U_W + \frac{h^2}{s^2}U_E$$

$$U_c = \frac{(r^2 + rh)U_N + (r^2 - rh)U_S + \frac{h^2}{s^2}U_W + \frac{h^2}{s^2}U_E}{2\left(r^2 + \frac{h^2}{s^2}\right)}$$

North: $\dfrac{r^2 + rh}{2\left(r^2 + \frac{h^2}{s^2}\right)} = \dfrac{r(r+h)}{2\left(r + \frac{h}{s}\right)\left(r - \frac{h}{s}\right)}$ $\qquad p_N = \dfrac{r^2 + rh}{2\left(r^2 + \frac{h^2}{s^2}\right)}$

South: $\dfrac{r^2 - rh}{2\left(r^2 + \frac{h^2}{s^2}\right)}$

East $\dfrac{h^2}{s^2}\,\dfrac{1}{2\left(r^2 + \frac{h^2}{s^2}\right)}$

West $\dfrac{h^2}{s^2}\,\dfrac{1}{2\left(r^2 + \frac{h^2}{s^2}\right)}$