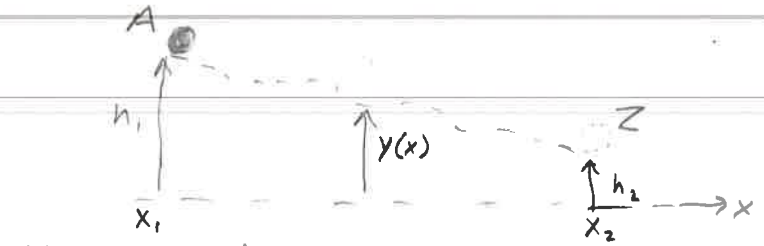
GES 554

Lesson 44

Calculus of Variations

Rolling ball on a track



Classical mechanics:

Energy = Potential Energy + Kinetic Energy

$$= mg(h_1 - y) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Velocity at $A = 0$

" " $Z = \sqrt{2g(h_1 - h_2)}$ not a function of $y(x)$

- What is the shortest <u>distance</u> between $A \to Z$?

  Shortest distance is a line. $y(x) = h_1 - \left(\frac{h_1 - h_2}{x_1 - x_2}\right)x$

- What is the shortest <u>time</u> between $A \to Z$?

  - $y'(x_1) = 0 \to t_{AZ} = \text{undefined}$, never starts moving.
  - $y' = \frac{-h_1 + h_2}{x_1 - x_2} \to$ constant acceleration $\to s = \frac{1}{2}at^2 \to t = \sqrt{\frac{2s}{a}}$

  gravity component in y dir is $= \frac{h_1 - h_2}{x_1 - x_2}$ and $s = \sqrt{(h_1 - h_2)^2 + (x_1 - x_2)^2}$

$$t = \sqrt{\frac{2\sqrt{(h_1 - h_2)^2 + (x_1 - x_2)^2}}{g\left(\frac{h_1 - h_2}{x_1 - x_2}\right)}}$$

$$\text{Average velocity} = \frac{1}{T}\int_0^T v\,dt = \frac{1}{T}\int_0^T at\,dt = \frac{g\left(\frac{h_1 - h_2}{x_1 - x_2}\right)}{T}\frac{T^2}{2}$$

$$= gT\left(\frac{h_1 - h_2}{x_1 - x_2}\right)\left(\frac{1}{2}\right) = \text{half final velocity}$$

Can we do better?

<u>yes</u>, Cycloid.

- Steep drop and horizontal
  - Obtains velocity of $\sqrt{2g(h_1-h_2)}$ and curves to horizontal

    Time required is $t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2(h_1-h_2)}{g}}$
  - Time required to move along horizontal is $t = \frac{D}{V} = \frac{x_2-x_1}{\sqrt{2g(h_1-h_2)}}$

$$t = \frac{(x_2-x_1)\sqrt{2(h_1-h_2)}}{2(h_1-h_2)} \frac{1}{\sqrt{g}}$$

- Total time

$$T = \sqrt{\frac{2(h_1-h_2)}{g}}\left(1 + \frac{1}{2}\frac{x_2-x_1}{h_1-h_2}\right)$$

- Avg Velocity

$$\bar{V} = \frac{1}{T}\int_0^T V\,dt = \frac{1}{T}\int_0^{\sqrt{\frac{2(h_1-h_2)}{g}}} \cancel{0}\,dt + \frac{1}{T}\int_0^{\sqrt{\frac{2(h_1-h_2)}{g}}\frac{1}{2}\frac{x_2-x_1}{h_1-h_2}} \sqrt{2g(h_1-h_2)}\,dt$$

$$= \frac{\sqrt{2g(h_1-h_2)}\sqrt{g}}{\sqrt{2(h_1-h_2)}} \frac{1}{1+\frac{1}{2}\left(\frac{x_2-x_1}{h_1-h_2}\right)} \sqrt{\frac{2(h_1-h_2)}{g}} \frac{1}{2}\left(\frac{x_2-x_1}{h_1-h_2}\right)$$

$$\bar{V} = \frac{\sqrt{2g(h_1-h_2)}\ \frac{1}{2}\left(\frac{x_2-x_1}{h_1-h_2}\right)}{1+\frac{1}{2}\left(\frac{x_2-x_1}{h_1-h_2}\right)} = (\text{Final Velocity}) \cdot \frac{\frac{1}{2}\frac{1}{\text{slope}}}{1+\frac{1}{2}\frac{1}{\text{slope}}}$$

$$= (FV) \cdot \frac{1}{2\,\text{slope} + 1}$$

Better results when average slope is small

vs

Calculus of Variations

Given a ~~functional~~ (function of a function), can you minimize $J(y)$?

$$J(y) = \int_a^b F(x,y,y')\,dx$$

We are looking for a function $y(x)$ that minimizes $J(y)$. What function(s) can we pick? Anything!

- Guess and check?! (Actually used in quantum physics. See Boyd 3.8)
  Not particularly efficient, but useful for upper / lower bounds.
- Calc' of Variations.

Derivation:

- Let $\bar{y}$ be the minimum function for $J(y)$. Thus $J(\bar{y} + \varepsilon\eta)$ is not the minimum.
- From calculus, find $\frac{dJ}{d\varepsilon}$ and set to 0. (classic optimization strategy)

$$\frac{d}{d\varepsilon} J(\bar{y}+\varepsilon\eta) = \frac{d}{d\varepsilon}\int_a^b F(\bar{y}+\varepsilon\eta)\,dx = 0$$

- Recall Leibnitz integral rule for derivatives of integrals

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} f_x(x,t)\,dt$$

Apply to our case $(x \to \varepsilon,\ t \to x)$

$$\frac{d}{d\varepsilon}\int_a^b F(x,y,y')dx = f(\varepsilon, b(\varepsilon))\,\cancel{b'(\varepsilon)}^{0!} - f(\varepsilon, a(\varepsilon))\,\cancel{a'(\varepsilon)}^{0} + \int_a^b f_\varepsilon(x,y,y')\,dx$$

$$= \int_a^b f_\varepsilon(x,y,y')\,dx$$

Total derivative of f_ε depends on x, y, y' since $J(x, y, y')$

$$\frac{df}{d\varepsilon} = \frac{df}{dx}\frac{dx}{d\varepsilon} + \frac{df}{dy}\frac{dy}{d\varepsilon} + \frac{df}{dy'}\frac{dy'}{d\varepsilon}$$

with $y = \bar{y} + \varepsilon\eta \Rightarrow \frac{dx}{d\varepsilon} = 0 , \quad \frac{dy}{d\varepsilon} = \eta$

and $\frac{dy}{dx} = \frac{d\bar{y}}{dx} + \frac{d}{dy}(\varepsilon\eta) \Rightarrow \frac{dy'}{d\varepsilon} = \underbrace{\frac{d}{d\varepsilon}\left(\frac{d\bar{y}}{dx}\right)}_{\to 0} + \cancel{\frac{d}{d\varepsilon}\frac{d}{dx}(\varepsilon\eta)} \; \frac{d}{dx}\underbrace{\frac{d}{d\varepsilon}(\varepsilon\eta)}_{\eta} = \frac{d\eta}{dx}$

$$\frac{df}{d\varepsilon} = 0 + \frac{df}{dy}\eta + \frac{df}{dy'}\frac{d\eta}{dx}$$

Subs,

$$\int_a^b \left(\frac{df}{dy}\eta + \underbrace{\frac{df}{dy'}\frac{d\eta}{dx}}_{\text{integrate by parts}}\right)dx = 0$$

integrate by parts $\int \phi \frac{d\psi}{dx}\,dx = -\int \frac{d\phi}{dx}\psi\,dx + \underbrace{\cancel{\phi\frac{d\psi}{dx}\Big|_a^b}}_{0}$ 0 since $\varepsilon\eta \to 0$ at boundary conditions

$$\int_a^b \left[\frac{df}{dy}\eta - \frac{d}{dx}\left(\frac{df}{dy'}\right)\eta\right]dx = 0$$

Pull η out

$$\int_a^b \left[\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right)\right]\eta\,dx = 0$$

Apply the fundamental lemma of the calculus of variations (i.e. $\eta \neq 0$)

translation: converts weak form to strong form

$$\boxed{\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0}$$

Euler - Lagrange Equ.

Example. $f(x, y, y') = y^2 + y'^2$

Minimize $J(y) = \int_0^1 (y^2 + y'^2)\,dx$

Apply Euler-Lagrange equation

$$\frac{dF}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0 \qquad \text{with } \frac{df}{dy} = 2y \quad \text{and} \quad \frac{df}{dy'} = 2y'$$

Subst

$$2y - \frac{d}{dx}(2y') = 0$$

Expand

$$2y - 2y'' = 0 \qquad \text{ODE}$$

Canonical form

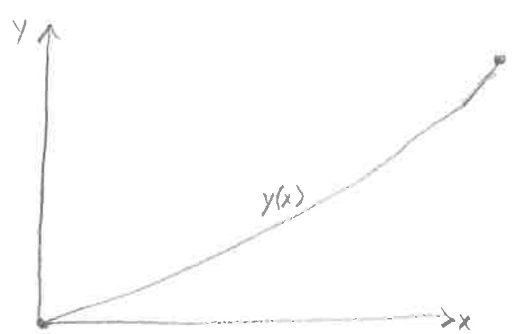
$$y'' - y = 0 \quad \Rightarrow \quad \begin{aligned} y &= Ae^{x} + Be^{-x} \\ y' &= Ae^{x} - Be^{-x} \\ y'' &= Ae^{x} + Be^{-x} \end{aligned}$$

$$y(0) = 0$$

$$y(1) = 1 \quad \Rightarrow \quad \begin{aligned} &y(0) = A\cancel{e^{0}}^{\,1} + B\cancel{e^{0}}^{\,1} = 0 = A + B \quad \Rightarrow A = -B \\ &y(1) = 1 = Ae^{1} + Be^{-1} = Ae^{1} - Ae^{-1} \end{aligned}$$

$$A = \frac{1}{e - e^{-1}} \approx 0.425$$

$$\boxed{y(x) = \frac{1}{e - e^{-1}} e^{x} - \frac{1}{e - e^{-1}} e^{-x}}$$



min $f = y^2 + y'^2$

$J(y) = \int_0^1 f\,dx$

What is the minimum value of J? $\qquad A = \dfrac{1}{e - e^{-1}}$

$$y(x) = Ae^{x} - Ae^{-x}$$

$$y'(x) = Ae^{x} + Ae^{-x}$$

$$y^2 = A^2 e^{2x} - A^2 \cancelto{1}{e^{x} e^{-x}} + A^2 e^{-x} e^{-x}$$

$$y'^2 = A^2 e^{2x} + A^2 \cancel{e^{x} e^{-x}} + A^2 e^{-x} e^{-x}$$

$$f = y^2 + y'^2 = 2A^2 e^{2x} + 2A^2 e^{-2x}$$

$$J = \int f \, dx = 2A^2 \int e^{2x} + e^{-2x} \, dx$$

$$\boxed{J_{min} = 2A^2\left(\frac{e^2}{2} + \frac{e^{-2}}{-2}\right) \approx 1.27}$$

Try a non-optimal $y(x)$

$y(x) = x$ $\quad \frac{dy}{dx} = 1$ $\quad f = y^2 + y'^2$

$= x^2 + 1$

$$J(y) = \int_0^1 f \, dx = \int_0^1 x^2 + 1 \, dx \qquad \frac{x^3}{3} + x \Big|_0^1$$

$$\frac{1}{3} + 1 = 1\tfrac{1}{3} = \boxed{4/3}$$

$J = 1.33 > 1.27$

Try another non-optimal $y(x)$

$y(x) = x^{10}$

$\frac{dy}{dx} = 10x^9$ $\quad f = x^{20} + 100\,x^{18}$

$J = 5.31 > 1.27$

$$\boxed{\text{You will never find } J < 1.27}$$

Brachistochrone (Cycloid) "Race to the bottom"

$$T = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{1+y'^2}{y}}\, dx \Rightarrow f = \frac{1}{\sqrt{2g}} \sqrt{\frac{1+y'^2}{y}}$$

you can see that the $\frac{1}{\sqrt{2g}}$ term doesn't matter.

Use Euler-Lagrange

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$f = \frac{1+y'^2}{y} = (1+y'^2)\,y^{-1}$$

$$\frac{df}{dy} = \frac{d}{dy}\left(\frac{1+y'^2}{y}\right)^{1/2} = \frac{d}{dy}\left(f^{1/2}\right)\frac{df}{dy} = \frac{1}{2} f^{-1/2} \frac{df}{dy} = \frac{1}{2} f^{-1/2} (-1)(1+y'^2)\,y^{-2}$$

$$= -\frac{1}{2}\left(\frac{y}{1+y'^2}\right)^{1/2}\left(1+y'^2\right)\frac{1}{y^2}$$

$$= -\frac{1}{2}\sqrt{\frac{1+y'^2}{y}}\,\frac{1}{y}$$

$$\frac{df}{dy'} = \frac{y'}{y'^2+1}\sqrt{\frac{1}{y}(y'^2+1)}$$

Subs.

$$-\frac{1}{2}\sqrt{\frac{1+y'^2}{y}}\,\frac{1}{y} - \frac{d}{dx}\left(\frac{y'}{y'^2+1}\sqrt{\frac{1}{y}\left(y'^2+1\right)}\right) = 0$$

This is a mess. Look for an easier approach!

Alternate Approach

We know that T is only a function of y and y' not x. $\left(f = \sqrt{\frac{1+y'^2}{2gy}}\right)$

- $$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} + \frac{df}{dy'}\frac{dy'}{dx} + \frac{df}{dx}$$

Solve for $\frac{df}{dy}\frac{dy}{dx} = \frac{df}{dx} - \frac{df}{dy'}\frac{dy'}{dx} - \frac{\partial f}{\partial x}$

- Multiply E-L by $\frac{dy}{dx}$

$$\underbrace{\frac{dy}{dx}\frac{df}{dy}}_{\text{subst.}} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$\frac{df}{dx} - \frac{df}{dy'}\frac{dy'}{dx} - \frac{\partial f}{\partial x} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

- $\frac{df}{dx} = 0$ (we said $T(y, y')$ only)

$$\frac{df}{dx} - \frac{df}{dy'}\frac{dy'}{\cancel{dx}} - \frac{df}{dx} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

This gives the Beltrami identity

$$f - y'\frac{\partial f}{\partial y'} = C$$

- Thus, we need to find

$$\frac{df}{dy'} = \frac{1}{2}\sqrt{\frac{2gy}{1+y'^2}} \cdot \frac{1}{2gy} \cdot 2y' = y'\sqrt{\frac{1}{1+y'^2}}\frac{1}{\sqrt{2gy}}$$

- Such that

$$\sqrt{\frac{1+y'^2}{2gy}} - y'^2\sqrt{\frac{1}{1+y'^2}}\sqrt{\frac{1}{2gy}} = C.$$

- Mult by $\sqrt{1+y'^2}\sqrt{2gy}$

$$1 + \cancel{y'^2 - y'^2}^{\,0} = C\sqrt{1+y'^2}\sqrt{2gy}$$

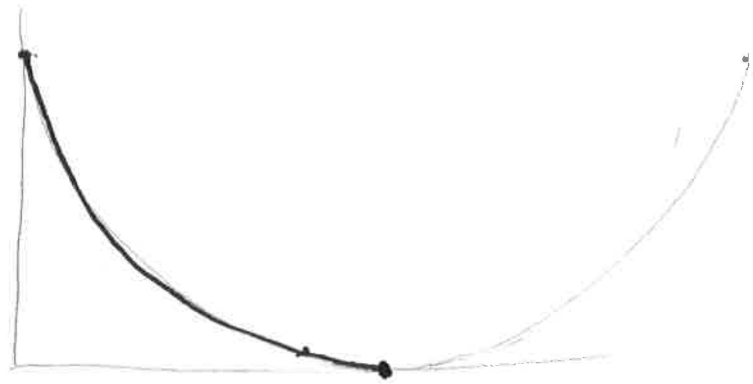
- Square

$$1 = C^2(1+y'^2)(2gy) \Rightarrow \frac{1}{C^2 2g} = (1+y'^2)y = C_2$$

Solution

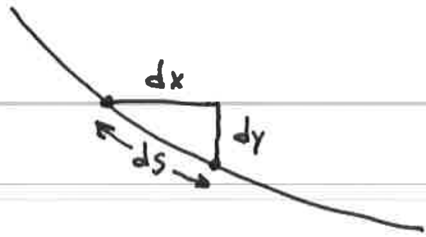
$$x = \frac{1}{2} C_2^2 (\xi - \sin \xi)$$

$$y = \frac{1}{2} C_2^2 (1 - \cos \xi)$$



Equal time <u>regardless</u> of starting point.

Derivation of ds on Farlow p 354



ds is along track

$$ds^2 = dx^2 + dy^2 \quad \Rightarrow \quad ds = \sqrt{dx^2 + dy^2}$$

pull out dx

$$dy = \frac{dy}{dx} dx \quad \Rightarrow \quad ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$\boxed{ds = dx\sqrt{1 + y'^2}}$$

Thus

$$T = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{1 + y'^2}{y}}\, dx$$

L44P21

$$J(y) = \int_0^1 \sqrt{1+y'^2}\, dx \qquad y(0) = 0$$
$$y(1) = 1$$

$$f = \sqrt{1+y'^2}$$

$$\frac{df}{dy} = 0 \qquad \frac{df}{dy'} = \frac{1}{2}\left(1+y'^2\right)^{-1/2} 2y'$$

Euler Lagrange

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$0 - \frac{d}{dx}\left(\frac{1}{2}\left(1+y'^2\right)^{-1/2} 2y'\right) = 0$$

Thus

$$\frac{1}{2}\left(1+y'^2\right)^{-1/2} 2y' = C$$

$$2y' = 2C\left(1+y'^2\right)^{1/2}$$

Square

$$4y'^2 = (2C)^2\left(1+y'^2\right) \Rightarrow 4y'^2 - (2C)^2 y'^2 = (2C)^2 \Rightarrow \left(4-(2C)^2\right)y'^2 = (2C)^2$$

Solve

$$y'^2 = \frac{(2C)^2}{4-(2C)^2} = C_2 \Rightarrow \text{Square root} \quad y' = \pm\sqrt{C_2} = C_3$$

$$\frac{dy}{dx} = C \Rightarrow y = Cx + d$$

BCs

By inspection

$$\boxed{y(x) = x}$$

L44P2

Minimize $J(y) = \frac{1}{2}\int_{t_1}^{t_2} (m\dot{y}^2 - ky^2)\, dt$

$f = m\dot{y}^2 - ky^2$ $\qquad$ $f(y, \dot{y})$ only

Euler-Lagrange (coordinates)

$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$ $\qquad$ $x \to t$

$$\boxed{\frac{df}{dy} - \frac{d}{dt}\left(\frac{df}{d\dot{y}}\right) = 0}$$

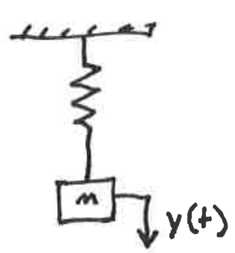
Find

$\frac{df}{dy} = -ky2$ $\qquad$ $\frac{df}{d\dot{y}} = 2m\dot{y} \Rightarrow \frac{d}{dt}\frac{df}{d\dot{y}} = 2m\ddot{y}$

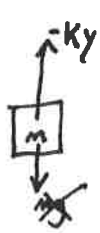
Subs

$-ky2 - 2m\ddot{y} = 0 \Rightarrow \boxed{m\ddot{y} + ky = 0}$ ✓

Verify



$F = ma$



$-ky = m\ddot{y} \Rightarrow \boxed{m\ddot{y} + ky = 0}$ ✓

L44P3

$$J(y) = \int_0^{\pi/2} \left(y'^2 - y^2\right) dx \qquad y(0) = 0, \quad y(\pi/2) = 1$$

Euler Lagrange

$$f = y'^2 - y^2 \qquad \frac{df}{dy} = -2y \qquad \frac{df}{dy'} = 2y'$$

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$-2y - \frac{d}{dx}\left(2y'\right) = 0$$

$$-2y - 2y'' = 0 \quad \Rightarrow \quad \underline{y'' + y = 0}$$

Solve ODE

$$y(x) = A\sin(x) + B\cos(x)$$

BCs

$$y(0) = 0 = A\,\cancel{\sin 0}^{\,0} + B\,\cancel{\cos 0}^{\,1} \quad \Rightarrow \quad B = 0$$

$$y(\tfrac{\pi}{2}) = 1 = A\sin(\tfrac{\pi}{2}) \quad \Rightarrow \quad A = 1$$

$$\boxed{\bar{y} = \sin(x)}$$