

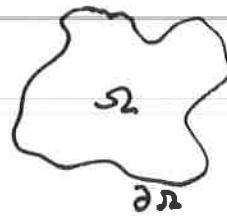
GES 554

Lesson 45

Ritz (theory for finite element derivation)

Method of Ritz

$$\text{Minimize } J(u) = \int_{\Omega} (U_x^2 + U_y^2 + 2uf) d\Omega$$



Solution to $U_{xx} + U_{yy} = f$ on Ω
 $u(\partial\Omega) = 0$ on $\partial\Omega$

Q) How do we know $U_x^2 + U_y^2 + 2uf$ corresponds to $U_{xx} + U_{yy} = f$?

A) $J(u)$ is the potential energy.

You can derive that $F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y} = 0$ (see L44 p4)

from the minimization of

$$J(u) = \iint F dx dy$$

Try it:

$$F_u = 2f$$

$$F_{u_x} = 2U_x$$

$$F_{u_y} = 2U_y \Rightarrow 2f - \frac{d}{dx} 2U_x - \frac{d}{dy} 2U_y = 0$$

divide by 2 and rearrange.

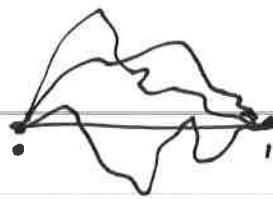
$$U_{xx} + U_{yy} = f$$

History

Ritz in 1D

$$U_{xx} = 1 \quad \text{on } 0 \leq x \leq 1$$

$U(0) = 0 \text{ and } U(1) = 0$



- We want to minimize $J(u) = \int_0^1 (U_x^2 + 2u) dx$
- We need an expansion for u that fits the BCs. "Basis functions"
 $\phi_1 = x(1-x)$
 $\phi_2 = x^2(1-x) \Rightarrow u = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$
 $\phi_3 = x^3(1-x) \quad U_x = \dots$

- Symbolic algebra/calc

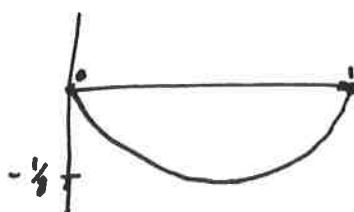
$$\begin{aligned} \frac{dJ}{da_1} &= \frac{2}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{5}a_3 + \frac{1}{3} \\ \frac{dJ}{da_2} &= \frac{1}{3}a_1 + \frac{4}{15}a_2 + \frac{1}{5}a_3 + \frac{1}{6} \\ \frac{dJ}{da_3} &= \frac{1}{5}a_1 + \frac{1}{5}a_2 + \frac{6}{35}a_3 + \frac{1}{10} \end{aligned} \quad \left. \right\} = 0$$

- Solve

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{4}{15} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{6}{35} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{6} \\ -\frac{1}{10} \end{pmatrix}$$

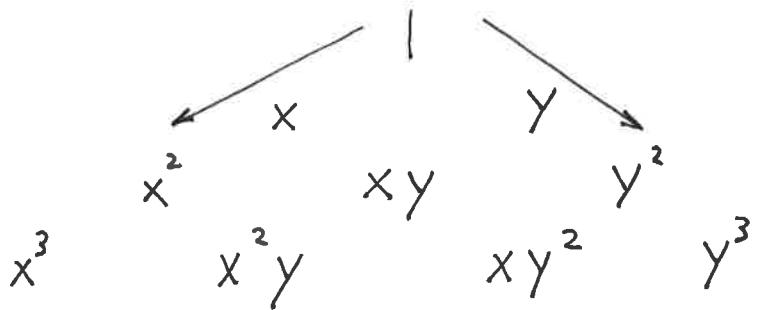
$$a_1 = -0.5 \quad a_2 = a_3 = 0$$

$$U(x) = -\frac{1}{2}x(1-x)$$



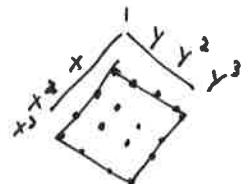
Generating basis functions

- Pascal triangle visualization



- Tensor product (fill rectangle of Pascal)

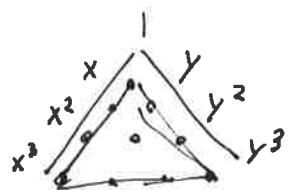
$$X = 1, x, x^2, x^3 \\ Y = 1, y, y^2, y^3 \Rightarrow \phi = X \cdot Y$$



$$\phi = 1, x, y, xy, xy^2, yx^2, x^2, y^2, xy^3, x^2y^2, x^2y^3, x^3, y^3 \\ x^3y, x^3y^2, x^3y^3, xy^3, x^2y^3, x^3y^3$$

Maximum order of ϕ is the order of $X \otimes$ order of Y .

- Serenicity (fill horizontal lines of Pascal)



$$\phi = 1, x, y, x^2, y^2, xy \\ x^3, x^2y, xy^2, y^3$$

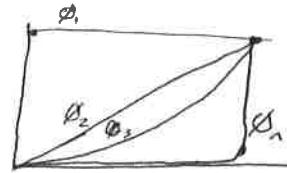
maximum order of ϕ
is the maximum order of X and Y

Ritz and Galerkin methods with higher order basis functions

$$\Phi = \text{set of functions} \quad \text{for} \quad \int_0^1 f(\phi) \phi_j dx \quad M_{ij} = \int_0^1 \phi_i \phi_j dx$$

- Polynomials

$$\Phi = \{1, x, x^2, x^3, x^4, x^5, \dots\}$$



$$M_{ij} = \int_0^1 \phi_i \phi_j dx = \int_0^1 x^i x^j dx = \int_0^1 x^{i+j} dx = \frac{x^{i+j+1}}{i+j} \Big|_0^1 = \frac{1}{i+j}$$

$$M_{ii} = \frac{1}{i+j}$$

$$M_{00} = 1$$

Inverting this matrix becomes difficult as ~~$n \gg 1$~~ or even $n \gg 1$

- Chebyshev polynomials

$$T_n(\cos \theta) = \cos(n\theta)$$

$$T_0 = 1 \quad T_1 = x$$

$$T_{n+1} = 2x T_n - T_{n-1}$$

Solves

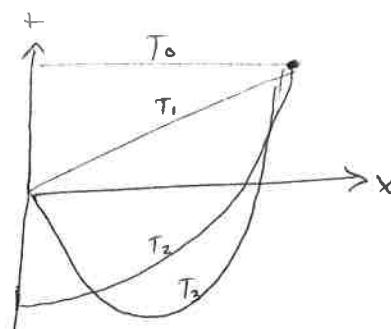
$$(1-x^2)y'' - xy' + n^2y = 0$$

$$T_0 = 1$$

$$T_1 = x$$

$$T_2 = 2x^2 - 1$$

$$T_3 = 4x^3 - 3x$$



Mass and stiffness matrices are well behaved.

Boyd says "If the domain is finite, but $u(x)$ is not a periodic function of x , Chebyshev polynomials are best"

"If the domain is periodic, use Fourier series"

Finite Elements in 1D

$$U_{xx} = 1 \quad \text{on } 0 \leq x \leq 1$$

$$U(0) = 0$$

$$U(1) = 0$$

Galerkin Approach

$$\int (U_{xx} - 1) \phi \, dx = 0$$

translate: reduce errors in gov eqn to zero with respect to basis/trial function.

- Basis functions

$$\phi_1 = x(1-x)$$

$$\phi_2 = x^2(1-x)$$

$$\phi_3 = x^3(1-x)$$

\Rightarrow

$$\phi_{1_{xx}} = -2$$

$$2-6x$$

$$6x - 12x^2$$

- Trial 1

$$\phi_1 = x - x^2$$

$$\int_0^1 (-2a_1 + (2-6x)a_2 + (6x-12x^2)a_3)(x - x^2) \, dx = 0 \Rightarrow -\frac{1}{3}a_1 - \frac{1}{6}a_2 - \frac{1}{10}a_3 - \frac{1}{6} = 0$$

- Trial 2

- Trial 3 similar

$$\begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{10} \\ -\frac{1}{6} & \frac{2}{15} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{3}{25} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{12} \\ \frac{1}{20} \end{pmatrix}$$

We already solved this for the Ritz problem....

Ritz and Galerkin are identical methods

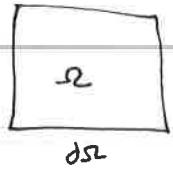
L45 P1

What is J for

$$U_{xx} + U_{yy} = 1$$

$$0 < x < 1 \quad 0 < y < 1$$

$$U = 0 \quad \text{on } \partial\Omega$$



A: The functional J of a square domain poisson PDE is

$$J(U) = \iint_{\Omega} (U_x^2 + U_y^2 + 2Uf) \, d\Omega$$

L45 P4

Show the E-L equation of

$$J(v) = \int_0^1 \int_0^1 (v_x^2 + v_y^2) dx dy$$

is

$$v_{xx} + v_{yy} = 0$$

A: From the notes, the multi-dimensional version of E-L is

$$\frac{dF}{du} - \frac{d}{dx}\left(\frac{dF}{dv_x}\right) - \frac{d}{dy}\left(\frac{dF}{dv_y}\right) - \frac{d}{dz}\left(\frac{dF}{dv_z}\right) = 0$$

$$F = v_x^2 + v_y^2$$

$$\frac{dF}{du} = 0, \quad \frac{dF}{dv_x} = 2v_x, \quad \frac{dF}{dv_y} = 2v_y$$

Substitute

$$0 - \frac{d}{dx}(2v_x) - \frac{d}{dy}(2v_y) = 0$$

$$-2v_{xx} - 2v_{yy} = 0$$

$$v_{xx} + v_{yy} = 0$$

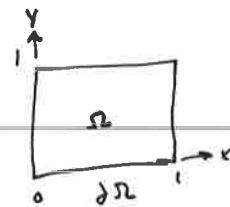


L45 P5

Find the energy of the solution to

$$U_{xx} + U_{yy} = \sin(\pi x)$$

$$U = 0 \text{ on } \partial\Omega$$



where the solution is

$$U(x,y) = \left(A e^{\pi y} + B e^{-\pi y} - \frac{1}{\pi^2} \right) \sin(\pi x)$$

A: From the notes,

$$F = U_x^2 + U_y^2 + 2Uf$$

$$\text{where } f = \sin(\pi x)$$

and $U(x,y)$ is given above

$$U_x = \left(A e^{\pi y} + B e^{-\pi y} - \frac{1}{\pi^2} \right) \pi \cos(\pi x)$$

$$U_y = \left(A \pi e^{\pi y} - B \pi e^{-\pi y} \right) \sin(\pi x)$$

$$2Uf = 2 \left(A e^{\pi y} + B e^{-\pi y} - \frac{1}{\pi^2} \right) \sin^2(\pi x)$$

Substitute

↓ some algebra

$$\begin{aligned} & \frac{1}{\pi^2} \left(\pi^4 (ae^{2\pi y} - b)^2 \sin^2(\pi x) + (\pi^2 ae^{2\pi y} + \pi^2 b - e^{\pi y})^2 \cos^2(\pi x) + \right. \\ & \left. + 2(\pi^2 ae^{2\pi y} + \pi^2 b - e^{\pi y}) e^{\pi y} \sin^2(\pi x) \right) e^{-2\pi y} \end{aligned}$$

Integrate $J = \iint_{\Omega} F d\Omega$ with a CAS

$$J = -\frac{\pi a^2}{2} + \frac{\pi a^2}{2} e^{2\pi} - \frac{\pi b^2}{2e^{2\pi}} + \frac{\pi b^2}{2} - \frac{1}{2\pi^2}$$