
GES 554
Lesson 4b
Perturbation Methods

Perturbation Methods for PDEs.

Q) How can you solve problems of the form (Farlow example): $\nabla^2 u + u^2 = 0$ or even
 $u(1, \theta) = \cos \theta$

A) Numerical solution. (One off problem)

A) Perturb a modified PDE (Multiple problems)

$$\nabla^2 u + \epsilon u^2 = 0$$

So we expect the solution to be (approximately) $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots$
 Plug u into PDE

$$\nabla^2(u_0 + \epsilon u_1 + \epsilon^2 u_2) + \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2)^2 = 0$$

$$u_0 + u_1 \epsilon + u_2 \epsilon^2 = \cos \theta$$

Match terms

$$\nabla^2 u_0 + \cancel{\epsilon} = 0 \quad \left. \begin{array}{l} \\ u_0 = \cos \theta \end{array} \right\} \text{we recovered the } \overset{\text{easy}}{\text{PDE}} \quad \checkmark$$

Solve for $u_0 \Rightarrow u_0(r, \theta) = r \cos \theta$

Match next set of terms

$$\nabla^2(\epsilon u_1) + \epsilon u_0^2 = 0 \quad \left. \begin{array}{l} \\ \epsilon u_1 = 0 \end{array} \right\} \text{This is a non-homogeneous PDE with terms from the previous matched terms.}$$

Solve for u_1 .

Strategy: Apply Taylor Series and Match terms of ϵ

1D example.

$$U_{xx} + U^2 = 0$$

$$U(0) = 1 \quad U(1) = 1$$



Approximate with $U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots$ from $U_{xx} + \epsilon U^2 = 0$

Subs.

$$\frac{d^2}{dx^2}(U_0 + \epsilon U_1 + \epsilon^2 U_2) + \epsilon(U_0 + \epsilon U_1 + \epsilon^2 U_2)^2 = 0$$

$$U_0 + \epsilon U_1 + \epsilon^2 U_2 \rightarrow \Rightarrow U_0(0) = 1$$

$$U_0(1) = 1$$

Solve

$$\frac{d^2 U_0}{dx^2} + \cancel{\epsilon U_0^2} = 0$$

$$U_0(0) = 1$$

$$U_0(1) = 1$$

$$\frac{d^2 U_0}{dx^2} + \cancel{\epsilon U_0^2} = 0$$

$$U_0(0) = 1$$

$$U_0(1) = 1$$

$$U_0 = Ax + B$$

apply BCs

$$\begin{cases} U_0 = 1 \\ U_0 = 1 \end{cases}$$

Match next set of terms.

$$\frac{d^2}{dx^2}(\epsilon U_1) + \epsilon U_0^2 = 0 \Rightarrow \frac{d^2 U_1}{dx^2} + U_0^2 = 0 \Rightarrow U_1 = -\frac{U_0 x^2}{2} + Ax + B$$

$$\epsilon U_1(0) = 0$$

$$\epsilon U_1(1) = 0$$

$$U_1(0) = 0$$

$$U_1(1) = 0$$

$$U_1 = -\frac{x^2}{2} + A \cdot 0 + B = 0$$

$$B = 0$$

Check if the solution is converged wrt U_n

$$U_{(2)} \approx U_0 + U_1 = 1 - \frac{x^2}{2} + \frac{1}{2}x$$

$$U_1 = -\frac{1}{2} + A + 0 = 0$$

$$A = \frac{1}{2}$$

Match next

$$U_1 = -\frac{x^2}{2} + \frac{1}{2}x$$

L46 P2

$$\nabla^2 U + U^2 = 0 \quad 0 < r < 1 \quad 0 \leq \theta < 2\pi$$

$$U(1, \theta) = \cos \theta$$

Perturbation

$$\nabla^2 U + \epsilon U^2 = 0$$

$$U(1, \theta) = \cos \theta$$

Power Series

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \dots$$

PDE

$$\nabla^2 (U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots) + \epsilon \underbrace{(U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots)^2}_{U_0^2 + \epsilon U_0 U_1 + \epsilon^2 U_0 U_2 + \dots + \epsilon U_0 U_1 + \epsilon^2 U_1^2 + \epsilon^2 U_2 U_0} = 0$$

Expand

$$\nabla^2 U_0 + \epsilon \nabla^2 U_1 + \epsilon^2 \nabla^2 U_2 + \epsilon U_0^2 + \epsilon^2 U_0 U_1 + \epsilon^3 (U_1^2 + 2U_0 U_2) + \dots = 0$$

Collect terms of $\epsilon, \epsilon^2, \epsilon^3, \dots$

- $\nabla^2 U_0 = 0$ with $U_0(1, \theta) = \cos \theta \Rightarrow U_0(r, \theta) = r \cos \theta$

- $\epsilon \nabla^2 U_1 + \epsilon U_0^2 = 0$ with $U_0(1, \theta) = 0$

$$\nabla^2 U_1 + \underbrace{r^2 \cos^2 \theta}_\text{still a linear problem!} = 0 \quad \stackrel{\text{or}}{\Rightarrow} \quad \nabla^2 U_1 + \frac{r^2}{2} + \frac{r^2}{2} \cos 2\theta = 0$$

$$U_1(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \ln\left(\frac{R}{r}\right) r^2 \cos^2 \theta \rho \, d\rho \, d\phi \quad \text{Tried with CAS... failed!}$$

- $\epsilon^2 \nabla^2 U_2 + \epsilon^2 U_0 U_1 = 0$ with $U_2(1, \theta) = 0$

$$\nabla^2 U_2 + 2U_0 U_1 = 0$$

Approximate solution: $U \approx U_0 + U_1 + U_2$

But wait ... there's more.

Green's functions failed, but we can brute force the solution to $\nabla^2 U_1 + r^2 \cos^2 \theta = 0$

$$\nabla^2 U_1 + r^2 \cos^2 \theta = 0 \Rightarrow \nabla^2 U_1 + \frac{r^2}{2} + \frac{r^2}{2} \cos(2\theta) = 0$$

pick a solution form

$$U_1 = A + Br + Cr \cos(2\theta) + Dr^2 + Er^2 \cos(2\theta) \\ + Fr^3 + Gr^3 \cos(2\theta) + Hr^4 + Ir^4 \cos(2\theta) + \dots$$

$$\nabla^2 U = U_{rr} = \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} \quad \text{substitute in the above} \uparrow$$

$$= 2D + 2E \cos(2\theta) + 6Fr + 6Gr \cos(2\theta) + 12Hr^2 + 12Ir^2 \cos(2\theta)$$

$$+ \frac{B}{r} + \frac{C}{r} \cos(2\theta) + 2D + 2E \cos(2\theta) + 3Fr + 3Gr \cos(2\theta)$$

$$+ 4Hr^2 + 4Ir^2 \cos(2\theta)$$

$$+ 4\left(\frac{C}{r} \cos(2\theta) - E \cos(2\theta) - 6r \cos(2\theta) - 4Ir^2 \cos(2\theta)\right)$$

$$= -\frac{r^2}{2} - \frac{r^2}{2} \cos(2\theta) \quad \leftarrow \text{from PDE to solve.}$$

Match terms

$$\bullet 12Hr^2 + 4Ir^2 = -\frac{r^2}{2} \Rightarrow 16H = -\frac{1}{2} \Rightarrow H = -\frac{1}{32}$$

$$\bullet 12Ir^2 \cos(2\theta) + 4Ir^2 \cos(2\theta) - 4Ir^2 \cos(2\theta) = -\frac{r^2}{2} \cos(2\theta) \Rightarrow I = -\frac{1}{24}$$

$$U_{1,\text{guess}} = -\frac{1}{32} r^4 - \frac{1}{24} r^4 \cos(2\theta)$$

$$U_1 = U_{\text{homogeneous}} + U_{\text{guess}} = 0 \text{ at } r=1$$

We know that $U_{\text{homogeneous}} = a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$

And with $U_1(1, \theta) = 0$, this gives $b_n = 0$, $a_0 \neq 0$ and all other $a_n = 0$
 $a_2 \neq 0$

Why?

$$\begin{aligned} U_1(1) &= a_0 + a_1 r^1 \cos(\theta) + a_2 r^2 \cos(2\theta) + a_3 r^3 \cos(3\theta) \\ &\quad + b_0 \cdot 0 + b_1 r^1 \sin(\theta) + b_2 r^2 \sin(2\theta) + b_3 r^3 \sin(3\theta) \\ &\quad - \frac{1}{32} r^4 - \frac{1}{24} r^4 \cos(2\theta) = 0 \end{aligned}$$

Match terms.

$$a_0 - \frac{1}{32} = 0 \quad \text{and} \quad a_2 \cos(2\theta) - \frac{1}{24} \cos(2\theta) = 0$$

Solution is:

$$U_1 = \frac{1}{32} + \frac{1}{24} r^2 \cos(2\theta) - \frac{1}{32} r^4 - \frac{1}{24} r^4 \cos(2\theta)$$

Approximate solution is. (2 term expansion)

$$U = U_0 + U_1 + U_2$$

$$= r \cos \theta + \frac{1}{32} + \frac{1}{24} r^2 \cos(2\theta) - \frac{1}{32} r^4 - \frac{1}{24} r^4 \cos(2\theta)$$

Notice a couple of issues.

- The 2 term solution doesn't quite fit the BC.

$$\frac{1}{32} \neq 0$$

However, the PDE is reasonably ok since $\left(\frac{1}{32}\right)^2$ is "small".

- The square term U^2 generates higher order terms in the solution

$$\nabla^2 U + U^2 = 0 \Rightarrow F, r^2, r^4$$

- The next perturbation step will be complicated !!

$$\nabla U_2 + 2 U_0 U_1 = 0$$

$$\nabla U_2 + \frac{2}{32} r \cos \theta + \frac{2}{24} r^3 \cos(2\theta) \overset{\cos(\theta)}{\checkmark} - \frac{2}{32} r^5 \cos(\theta) - \frac{2}{24} r^5 \cos(2\theta) \cos(\theta) = 0$$

However, this step's PDE is still linear !

Perturbation converts non-linear PDEs into more complicated series solutions and of linear PDEs