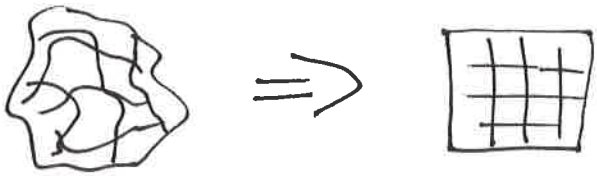
GES 554

Lesson 47

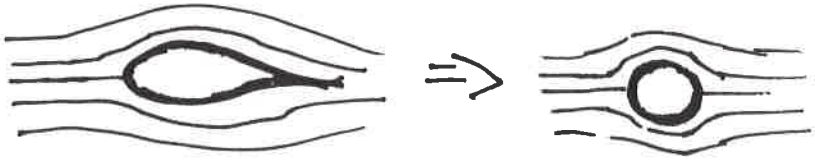
Conformal Mapping

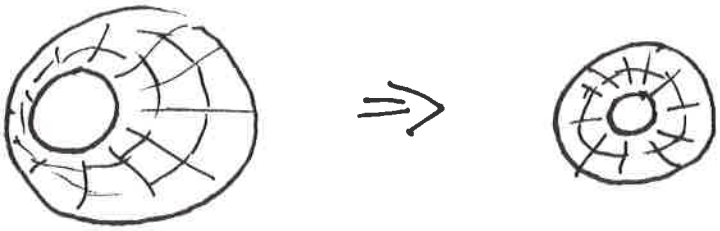
Conformal Mapping



Difficult, complex domain

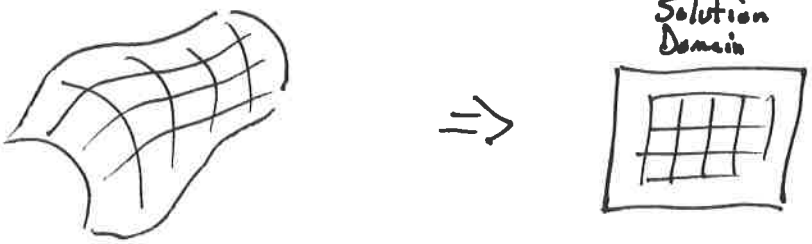
easy, regular domain





Arbitrary Validation Domain

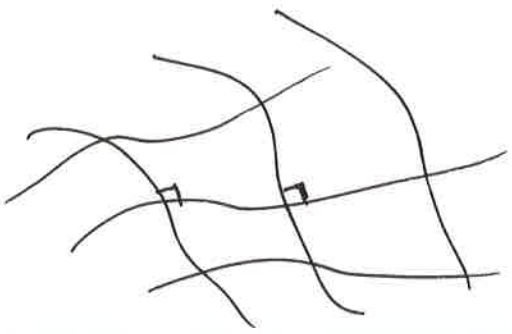
Manufactured Solution Domain



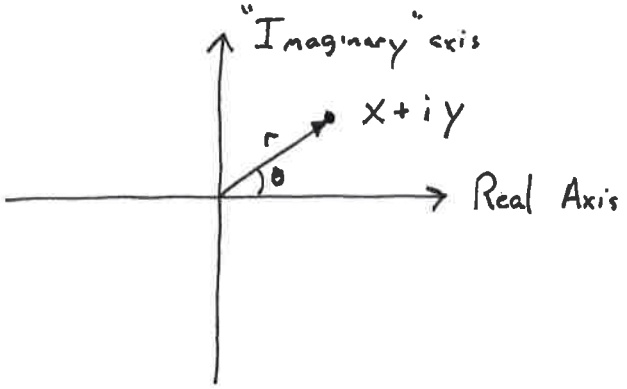
Apply a map from z to w:

$$w = f(z)$$

where $z = x + iy$



Complex #s



$$z = x + iy$$
$$= r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg(z) = \operatorname{atan}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Ex:

$$z = \sqrt{2}\, e^{i\pi/4}$$

~~$r = |z| = \sqrt{x^2+y^2}$~~

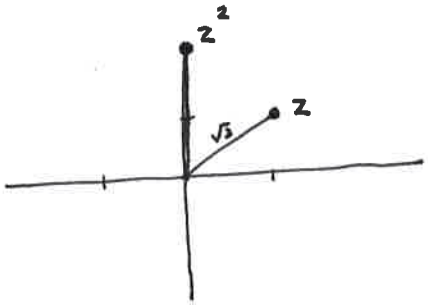
$$x = \sqrt{2}\cos\pi/4 = 1$$
$$y = \sqrt{2}\sin\pi/4 = 1$$

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\theta = \operatorname{atan}\left(\frac{1}{1}\right) = \operatorname{atan}(1) = \pi/4$$

Ex:

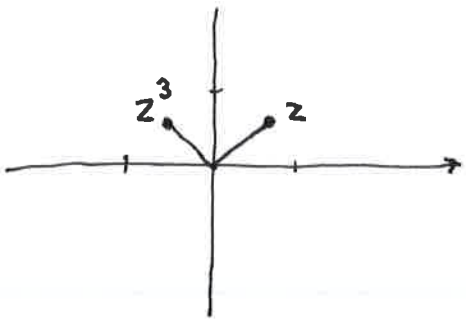
$$z = \sqrt{2}\, e^{i\pi/4} \qquad w = z^2$$

$$w = z^2 = \sqrt{2}^2\left(e^{i\pi/4}\right)^2 = 2e^{i\pi/2}$$

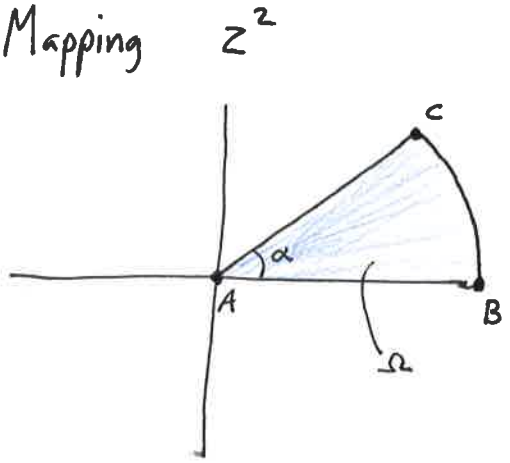


Ex: $z = 1\, e^{i\pi/4}$ $\qquad w = z^3$

$$w = e^{i3\pi/4}$$

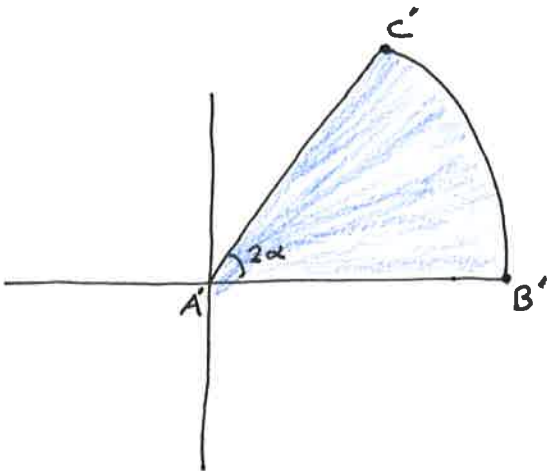


Mapping z^2



$\omega = z^2$

$\Rightarrow$



$$Z = X + iy \quad \Rightarrow$$

$$\omega = z^2 = (X+iy)^2$$

$$= x^2 + 2ixy - y^2$$

$$= (x^2 - y^2) + i(2xy)$$

$$Z = r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \text{atan}\left(\frac{y}{x}\right)$$

$$\omega = z^2 = r^2 e^{i2\theta}$$

$$r = |z| = \sqrt{(x^2-y^2)^2 + (2xy)^2}$$

$$= x^2 + y^2$$

$$\theta = \text{atan}\left(\frac{2xy}{x^2-y^2}\right)$$

$$A' = A^2 = (0+i0)^2 = 0$$

$$B' = B^2 = (b+i0)^2 = b^2 + i0$$

$$C' = C^2 = (c_r + i c_i)^2 = (c_r^2 - c_i^2) + i(2c_r c_i)$$

Analytic functions

$$f(z) = f(x+iy) \quad \Rightarrow \quad z = x + iy$$

derivatives

$$\frac{df}{dx} = \frac{df}{dz}\frac{dz}{dx}^{1} = \frac{df}{dz}$$

$$\frac{df}{dy} = \frac{df}{dz}\frac{dz}{dy}^{i} = i\frac{df}{dz}$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dz}\left(\frac{df}{dx}^{df/dz}\right)\frac{dz}{dx}^{1} = \frac{d}{dz}\left(\frac{df}{dz}\right) = \frac{d^2f}{dz^2}$$

$$\frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{dz}\left(\frac{df}{dy}\right)\frac{dz}{dy}^{i} = \frac{d}{dz}\left(i\frac{df}{dz}\right)i = i^2\frac{d^2f}{dz^2} = -\frac{d^2f}{dz^2}$$

Laplaces Egu.

$$\nabla^2 U = 0 = \frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} =$$

substitute above

$$0 \stackrel{?}{=} \frac{d^2U}{dx^2}^{d^2f/dz^2} + \frac{d^2U}{dy^2}^{-d^2f/dz^2} = 0 \quad \checkmark$$

$Z = x + iy$ satisfies $\nabla^2 U = 0$

In fact, any $z = (x+iy)^n$ satisfies Laplaces egu.

We call any $f(z)$ that can be written as $(x+iy)^n$ and converges an

Analytic function.

Cauchy - Riemann

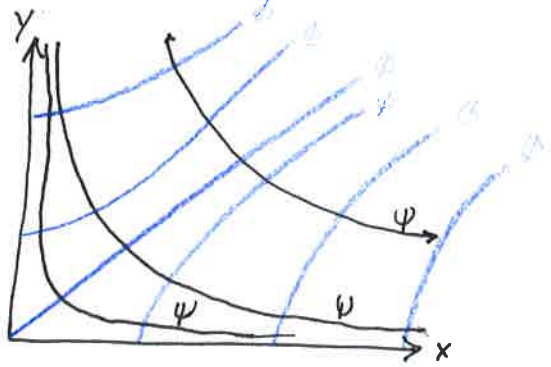
If a function $f(x+iy)$ converts into $u(x,y) + i\,s(x,y)$,

the function is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}$$

Warning!! Not all aero textbook provide consistent ψ and ϕ! Ex: $\phi_{aero} = -\phi_{math}$

For example, 90° corner flow



$$\psi(x,y) = A\,x\,y$$

$$\phi(x,y) = -\frac{1}{2}A\left(x^2 - y^2\right)$$

↑ My aero book skipped the minus!

This should (and does) satisfy $\nabla^2\psi = 0$ and $\nabla^2\phi = 0$

Use C-R to verify analytic...

$$f(z) = f(x+iy) = \overbrace{\psi(x,y)}^{u(x,y)} + i\,\overbrace{\phi(x,y)}^{s(x,y)}$$

$$= Axy + i\,\frac{-1}{2}A\left(x^2 - y^2\right)$$

Test

$$\frac{\partial u}{\partial x} \stackrel{?}{=} \frac{\partial s}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial \psi}{\partial x} = Ay \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{\partial \phi}{\partial y} = -\frac{1}{2}A(-2y)$$

$$\text{and} \quad Ay = Ay \quad \checkmark$$

$$\frac{\partial u}{\partial y} \stackrel{?}{=} -\frac{\partial s}{\partial x} \Rightarrow Ax = -\left(\frac{-1}{2}\right)A\,2x \quad \checkmark$$

L47p4

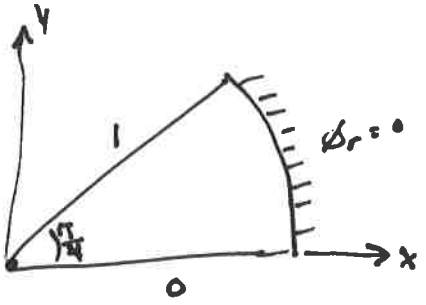
$\nabla^2 \phi = 0$

$\phi(r,0) = 0$

$\phi(r, \frac{\pi}{4}) = 1$

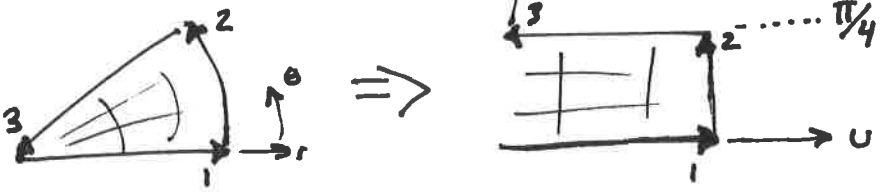
$\phi_r(1,\theta) = 0$

$0 < r < 1$

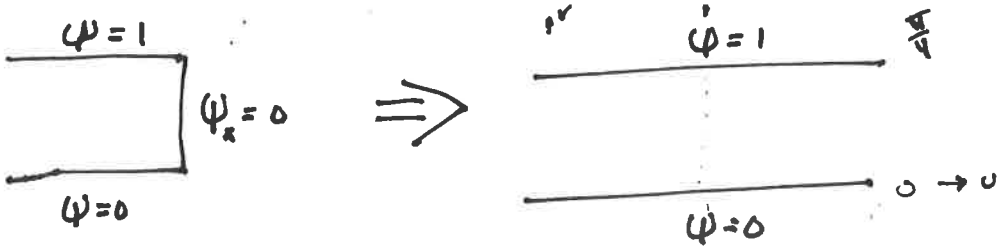


Map $w = \log|z| + i \arg(z) = \log(z)$

$= \log|x+iy| + i \arg(x+iy) = u + iv$



Find solution to $\nabla^2 \psi$ in the new domain



$\psi(u,v) = \frac{4}{\pi} v$ with $v = \arg(x+iy) = \theta$

Convert back to ϕ

- By inspection we notice that ψ is only a function of v.

$$\boxed{\phi = \frac{4}{\pi}\theta}$$

- solve for r, θ in terms of u, v.

$u + iv = \log|z| + i \arg(z)$

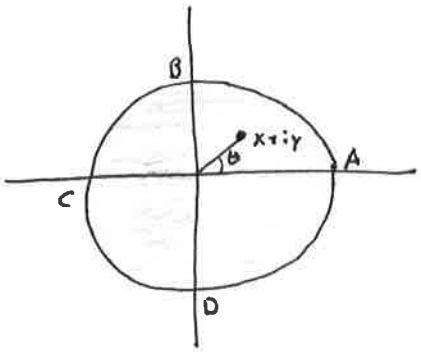
$= \log|z| + i\theta$

$= \log(r) + i\theta$

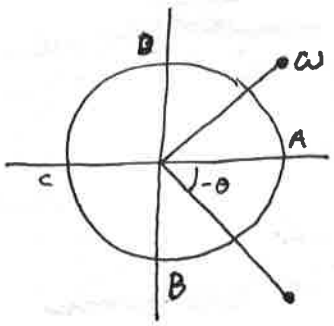
$\Rightarrow$ $u = \log(r)$

$v = \theta$

Mapping $\omega = \frac{1}{z}$



$\omega = \frac{1}{z}$



$z = x + iy$

$= r\cos\theta + i\, r\sin\theta$

$z = re^{i\theta}$

$\omega = \frac{1}{z} = \frac{1}{x+iy}$

multiply + divide by conjugate

$\omega = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$

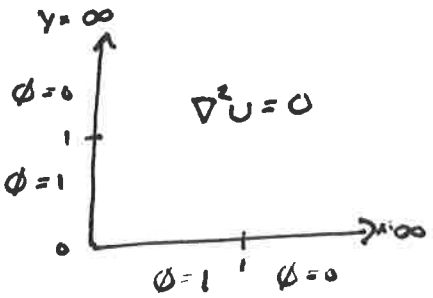
$= \frac{x}{r^2} - i\frac{y}{r^2}$ coordinate mixed system

$= \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$

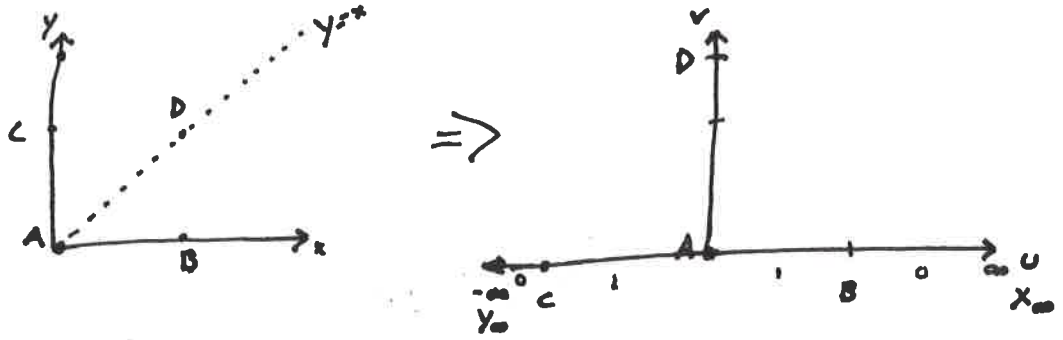
$\omega = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

Ex:

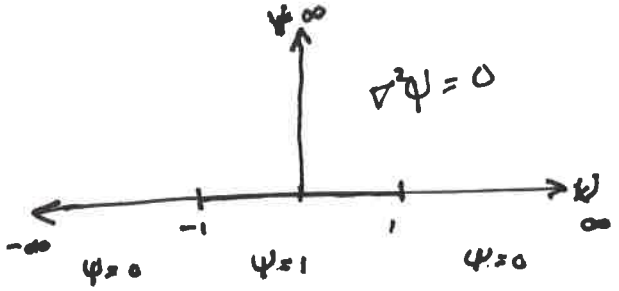
L47P3



Map $\omega = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = \underset{u}{(x^2-y^2)} + i\,\underset{v}{2xy}$

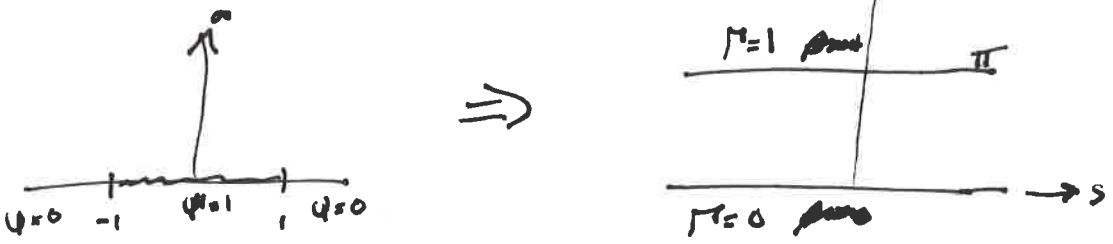


Solve new problem



$\psi(\omega) =$ This is a tough problem. I tried and failed with Fourier transforms.....

Map to a 3rd problem



$$\omega = \log\left(\frac{z-1}{z+1}\right)$$

Solution is $\Gamma(s,t) = \frac{t}{\pi}$

$$= \frac{u + iv - 1}{u + iv + 1} \qquad A + iB$$

$$= \frac{(u-1) + iv}{(u+1) + iv}$$

$$= \frac{((u-1) + iv)((u+1) - iv)}{(u+1)^2 + v^2}$$

$$= \frac{(u-1)(u+1) - iv(u-1) + iv(u+1) + v^2}{(u+1)^2 + v^2}$$

$$= (u^2 - 1 + v^2) + i(2v)$$

$$w = \log\left|\frac{z-1}{z+1}\right| + i\arg\left(\frac{z-1}{z+1}\right) = s + it$$

We know that Γ is only a function of t.

$$t = \arg\left(\frac{z-1}{z+1}\right) = \arg\left(\frac{u+iv-1}{u+iv+1}\right)$$

$$= \arg\left(\frac{u^2+v^2-1+2iv}{u^2+v^2+2u+1}\right) \qquad \text{some algebra.....}$$

$$= \operatorname{atan}\left(\frac{2v}{u^2+v^2-1}\right) \qquad \text{since} \quad \arg = \operatorname{atan}\left(\frac{y}{x}\right)$$

Complete Solution

$$\Gamma = \frac{t}{\pi} = \frac{1}{\pi}\operatorname{atan}\left(\frac{2v}{u^2+v^2-1}\right) = \psi$$

now subst u and v. $\qquad u = x^2-y^2$ and $v = 2xy$

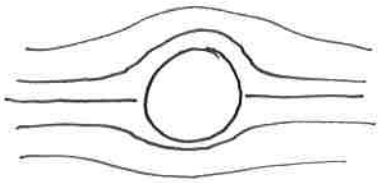
$$\phi = \frac{1}{\pi}\operatorname{atan}\left(\frac{4xy}{(x^2-y^2)^2+4x^2y^2-1}\right)$$

This is nothing short of amazing.

No Fourier transforms. No integrals.

2 Maps and some algebra.

Joukowski Transform

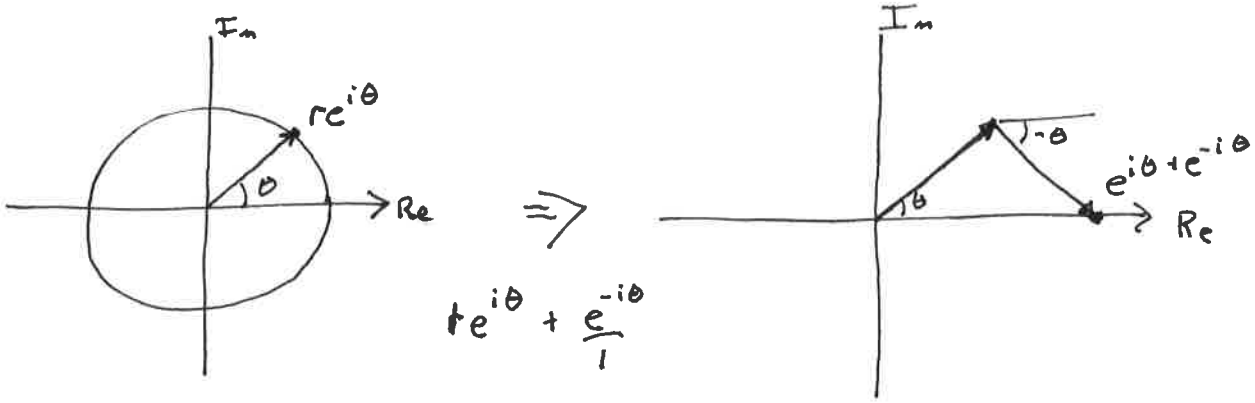


$$w = z + \frac{c^2}{z}$$

if $z = re^{i\theta}$ then $w = re^{i\theta} + \frac{c^2}{re^{i\theta}}$

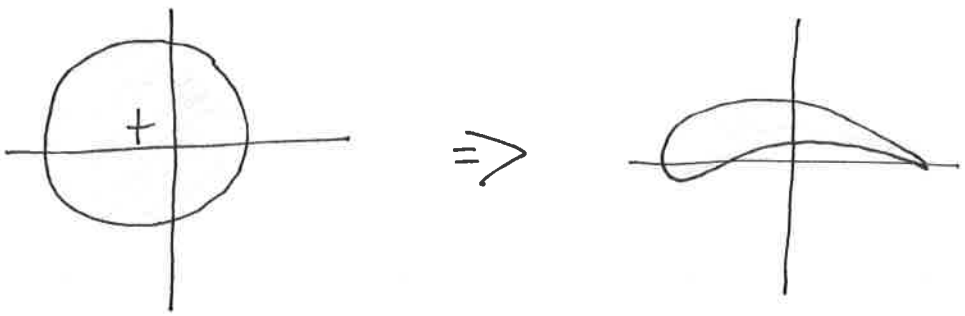
$$= re^{i\theta} + \frac{c^2 e^{-i\theta}}{r}$$

On a surface of radius $r = 1$ with $c^2 = 1$



$c^2 = 1$ transforms the circle to a line segment along the x-axis.

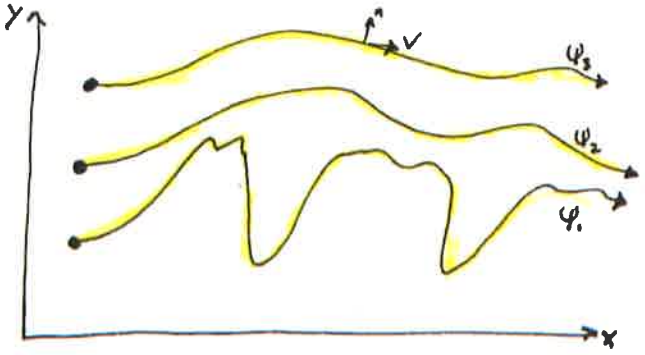
The Joukowski transform generates thickness (c^2) and camber (circle's center)



Congratulations

We finished the book

Stream function

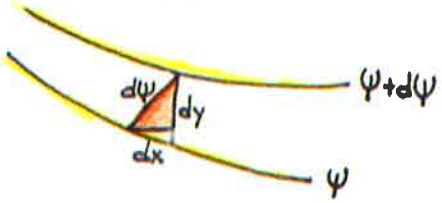


Streamlines are defined as lines which no fluid crosses. ($V \cdot n = 0$)

The difference in magnitude between two stream functions provides a measure of the volume of fluid flowing between the two streamlines per second (per unit depth).

ψ [ft²/s]

Pick a control volume between two streamlines (ψ and $\psi + d\psi$)



Flow into CV = $d\psi$

Flow out of CV = $u\,dy - v\,dx$

↖ negative since normal for dx side is $-\hat{j}$

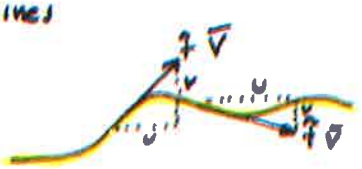
Definition of an exact differential of ψ

$$d\psi = \frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy$$

Match terms

$$\underbrace{\frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy}_{d\psi} = \underbrace{u\,dy - v\,dx}_{d\psi} \Rightarrow u = \frac{d\psi}{dy} \text{ and } v = -\frac{d\psi}{dx}$$

Streamlines



"The streamline is a curve whose tangent at every point coincides with the direction of the velocity vector"
Bertin-Smith

$$\frac{dx}{dy} = \frac{u}{v} \Rightarrow \underbrace{u\,dy - v\,dx = 0}_{\text{we saw this term above} = d\psi}$$

Thus along a streamline, no flow crosses the curve.

Div and Curl (incompressible, irrotational)

$$\nabla \cdot V \overset{?}{=} 0 \Rightarrow \frac{du}{dx} + \frac{dv}{dy} = \frac{d^2\psi}{dxdy} - \frac{d^2\psi}{dydx} = 0 \checkmark$$

$$\nabla \times V \overset{?}{=} 0 \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ u & v & w \end{vmatrix} = \frac{dv}{dx} - \frac{du}{dy} = \frac{d}{dx}\left(-\frac{d\psi}{dx}\right) - \frac{d}{dy}\left(\frac{d\psi}{dy}\right) = \nabla^2\psi = 0$$

ψ is also a Laplacian for incomp'e irrot'l

In fact, $F = \phi + i\psi$

$\nabla^2 F = 0$

Joukowski Transform (circle to airfoil and back)

We want but potential flow theory gives us circles

Remember that both ϕ and ψ were shown to satisfy $\nabla^2 F = 0$ where $F = \phi + i\psi$

Conformal Transforms with $\nabla^2 F = 0$

Complex # theory shows that if $\nabla^2 F = 0$, then any analytic function mapping also has $\nabla^2 F = 0$

Ex:

$f(z) = f(x+iy) \Rightarrow z = x+iy$

$$\frac{df}{dx} = \frac{df}{dz}\frac{dz}{dx} = \frac{df}{dz}\cdot 1 = \frac{df}{dz} \quad \text{and} \quad \frac{df}{dy} = \frac{df}{dz}\frac{dz}{dy} = i\frac{df}{dz}$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dz}\left(\frac{df}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{df}{dz}\right) = \frac{d^2 f}{dz^2}$$

$$\frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{dz}\left(\frac{df}{dy}\right)\frac{dz}{dy} = \frac{d}{dz}\left(i\frac{df}{dz}\right)i = i^2\frac{d^2 f}{dz^2} = -\frac{d^2 f}{dz^2}$$

Apply to Laplace eqn

$$\nabla^2 F = 0 = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{d^2 f}{dz^2} - \frac{d^2 f}{dz^2} = 0 \quad \checkmark$$

In fact, any analytic function mapping still satisfies $\nabla^2 = 0$

$z = (x+iy)^n$

Q: Can we find a complex function that transforms a circle to an airfoil?

Yes

and it is simple

$$\boxed{J(z) = z + \frac{c_1^2}{z}}$$ Joukowski Transform

And in fact, there are many discovered mapping functions

Joukowski (cont)

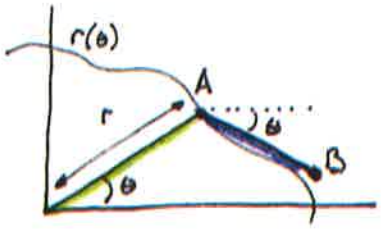
Remember the properties of complex #s

$$z = x + iy = re^{i\theta} \qquad \text{thus} \quad \frac{1}{z} = \frac{1}{|z|}e^{-i\theta} = \frac{1}{r}e^{-i\theta}$$

The map is $J(z) = z + c_1^2/z$

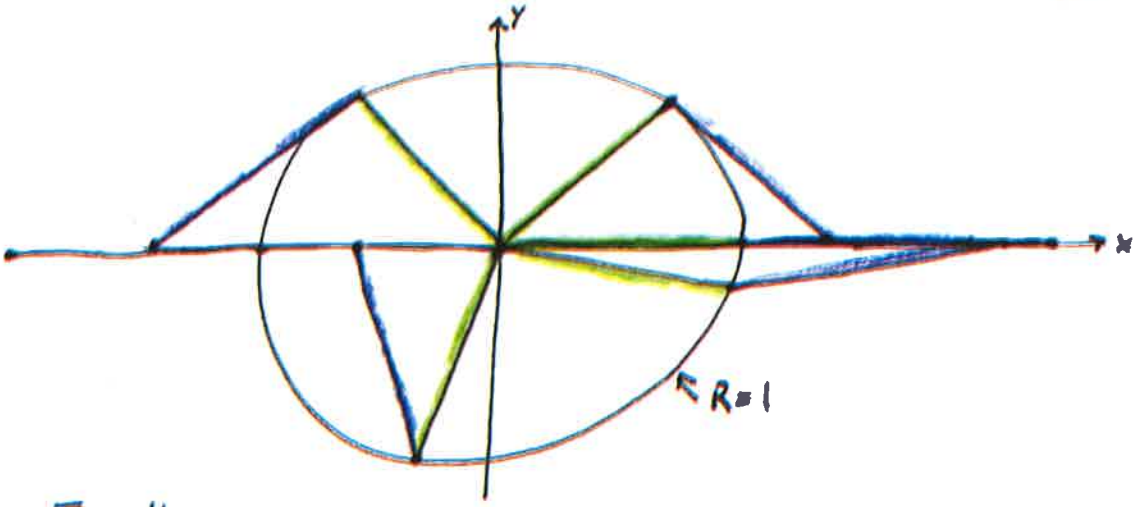
$$J(z) = \underbrace{re^{i\theta}}_{A} + \underbrace{\frac{c_1^2}{r}e^{-i\theta}}_{B}$$

Think of the transform as having two arms or vectors. Arm A traces the unmapped curve (a circle for us). Arm B starts at the end of A, is scaled by c_1^2/r, and is oriented in the conjugate as A's angle.



Ex:

Apply $J(z) = z + \frac{1}{z}$ to a unit circle at $x, y = 0, 0$ $\quad \frac{c_1^2}{r} = 1$



From this, we conclude that $z + \frac{1}{z}$ maps a circle at the origin to an infinitesimally thin plate with twice the chord

Varying C_1^2 affects the transformed airfoil's thickness (and chord)

look at $\theta = 90°$

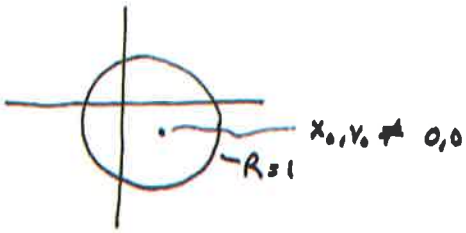
$= \uparrow + \downarrow = \Gamma_+ - \frac{C_1^2}{\Gamma_+}$

$= \downarrow + \uparrow = \Gamma_- + \frac{C_1^2}{\Gamma_-}$

$\Rightarrow$ thickness $= \Gamma_+ - \Gamma_-$

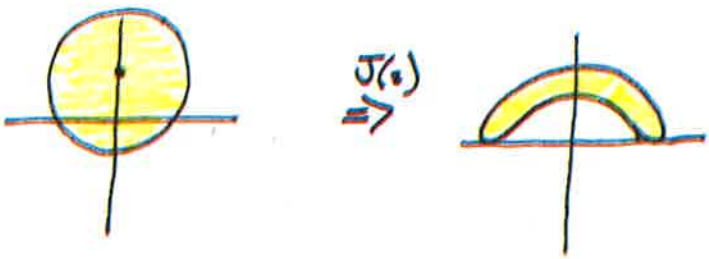
$- C_1^2 \left(\frac{1}{\Gamma_+} + \right.$

Also, the center of the circle can be offset



- Horizontal offset generates distinctive leading and trailing edges (when $C_1^2 < 1$)

- Vertical offset generates camber



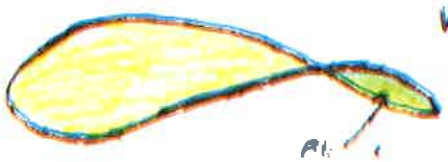
- Combinations generate cambered airfoils

$X, Y = -0.16, 0.23$

$C_1 = 0.8$

Q: Could we call this a J162380?

A: I've never heard seen this terminology!



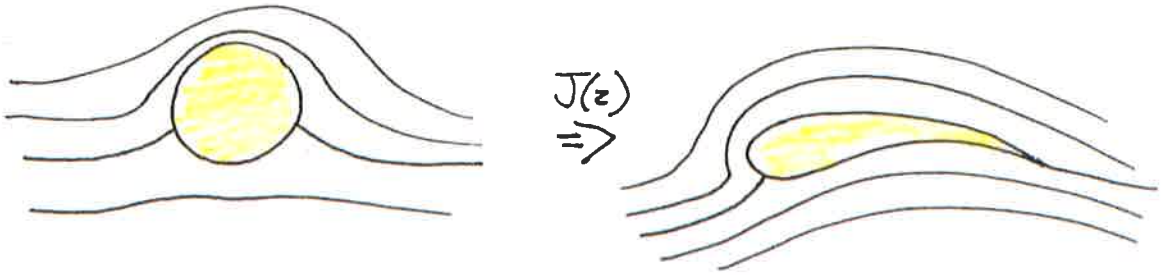
Warning!! Not every combination gives a valid airfoil

$X, Y = -0.32, 0.23$

$C_1 = 0.8$

Cylinder with Circulation

Now, we have the proper motivation to re-study the circle - potential flow.

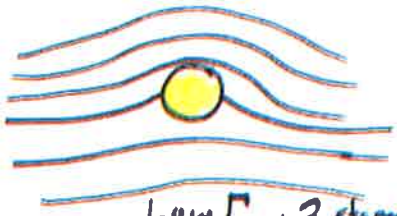


Remember from fluids,

$$\psi = +V\left(r - \frac{R^2}{r}\right)\sin\theta$$

$$= \underbrace{+Vr\sin\theta}_{\text{Uniform Flow}} \underbrace{- V\frac{R^2}{r}\sin\theta}_{\text{Doublet}}$$

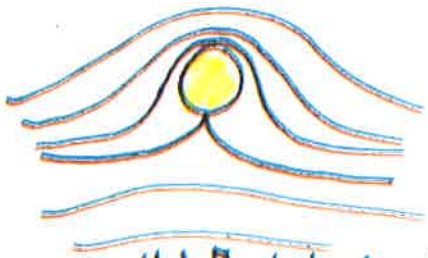
Adding a vortex at the center of the circle gives



$$\psi = Vr\sin\theta - V\frac{R^2}{r}\sin\theta + \underbrace{\frac{\Gamma}{2\pi r}}_{\text{vortex}}$$

Low Γ, 2 stagnation pts.

Remember that adding a vortex is acceptable since the domain we want can be isolated from the vortex with a branch cut.

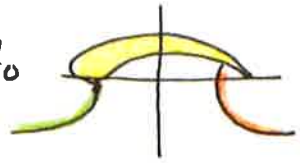


High Γ, 1 stagnation pt
($\Gamma = 4\pi V_\infty$)

Super high Γ, no stagnation pt on body.

So, which Γ do we pick?

How much Γ ??

We have but $J(z)$ transforms to 

To satisfy the Kutta Joukowski condition, the aft stagnation streamline must align to the trailing edge.

Why?

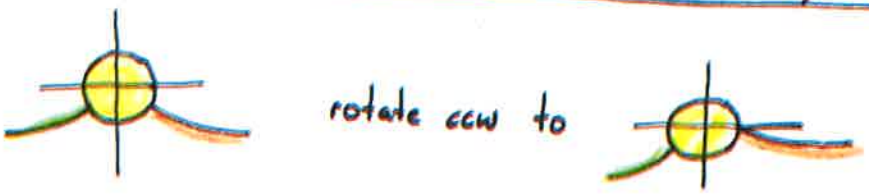
Real fluids have viscosity. If the TE is not a stagnation point, then the sharp TE generates infinite velocity/acceleration.

$\frac{V^2}{R} = \frac{V}{\frac{1}{\infty}}$ undefined.

A real fluid would have a separation point instantly, thus bringing the streamline back to the TE, since ω convects downstream.

Plus, K-J is what we observe in the wind-tunnel. Almost....

So, to align the TE, which is the location where the unit circle crosses the x-axis, with the stagnation point, we rotate the entire cylinder solution.



How much Γ again ....

What if we have a viscous case where the effective flow angle does not line up with the geometry?

Case: NACA 0070 (70% thick "strut")

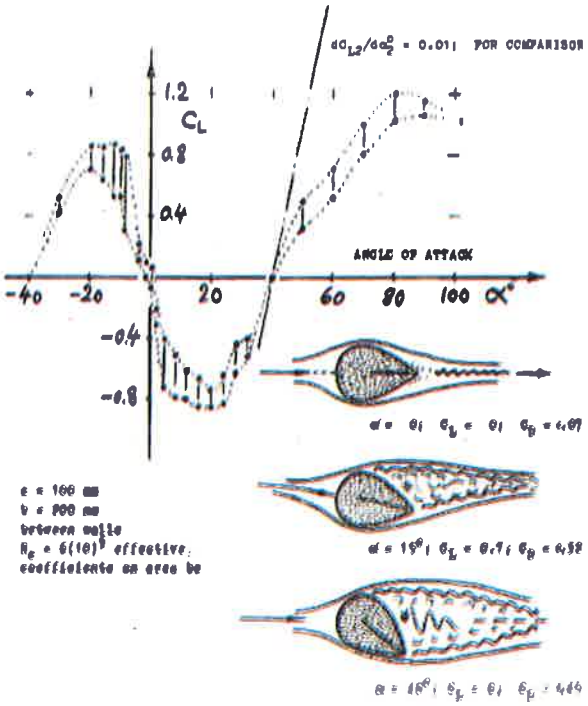
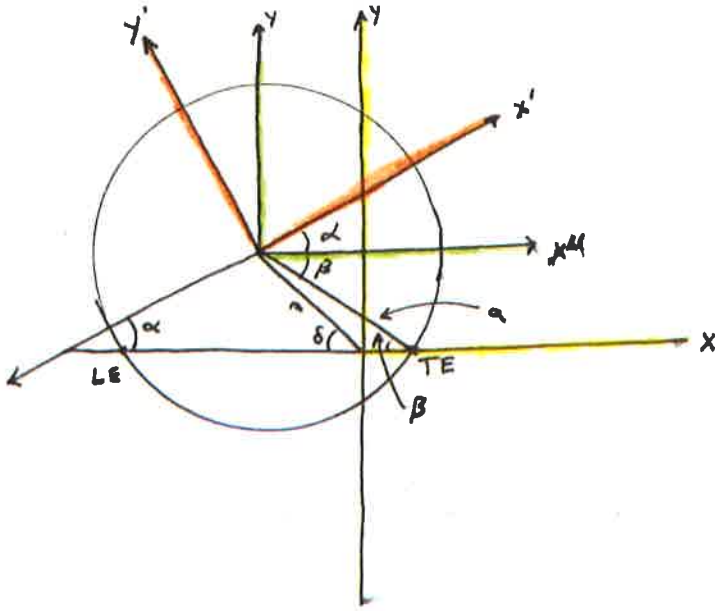


Figure 16. Lift of the extremely thick foil or strut section 0070, tested (18) to an angle of 90°. The lower and upper points plotted, correspond to the time-dependent fluctuations of the separated flow pattern. R'number above critical.

STRUT SECTION. "Streamline" sections, suitable to be applied in struts or in propeller blades near the hub, have usually higher thickness ratios than conventional foil sections. As an extreme example, the lift coefficient of an 0070 section (18) is presented in figure 16. The lift-curve slope at small angles of attack is strongly negative (!) up to $\alpha \approx 20^\circ$. The flow pattern proves that negative lift is the result of flow separation from the upper side of the section. As a consequence of the well-attached flow along the negatively cambered lower side of the section, suction develops there, thus producing negative lift. As the angle of attack is increased, the lower side eventually produces predominantly positive pressures and a correspondingly positive lift. Also note that the maximum lift coefficient at $\alpha = 90^\circ$, fluctuating between 1.0 and 1.2, corresponds to suction forces developing around the section's nose. - It is shown in (22,b) how the lift function of an airfoil with t/c = 66%, is almost perfectly linear, with a blunt trailing edge (see later). It can also be found in Chapter III of "Fluid-Dynamic Drag," that the

C_{l_α} is negative for $|\alpha| < 20^\circ$!!

From Hoerner's Fluid Dynamic Lift 2-9



$z' = r' e^{i\theta'}$

$z'' = z' e^{i\alpha}$ since $z'' = r' e^{i(\beta' + \alpha)} = r' e^{i\theta'} e^{i\alpha}$

$z = z'' - m e^{-i\delta} = z' e^{\alpha} - m e^{-i\delta}$

How much Γ? Enough to make the TE a stagnation point (i.e. $V = 0$)

$$w = -V\left(z' + \frac{R^2}{z'}\right) - \frac{i\Gamma}{2\pi} \ln \frac{z'}{R}$$

$$\frac{dw}{dz} = -V\left(1 - \frac{R^2}{z'^2}\right) - \frac{i\Gamma}{2\pi z'}$$

and the velocity magnitud $|q|$ is $|q| = \left|\frac{dw}{dz}\right|$

Thus, at the TE $= -V + \frac{VR^2}{z'^2_{TE}} - \frac{i\Gamma}{2\pi z'_{TE}}$

$$z'_{TE} = a e^{-i(\alpha+\beta)} \qquad (z'_{TE})^2 = a^2 e^{-2i(\alpha+\beta)}$$

$$-V + \frac{VR^2}{a^2} e^{2i(\alpha+\beta)} - \frac{i\Gamma}{2\pi a} e^{-i(\alpha+\beta)} = 0 \qquad \text{also } a = R$$

Solve for Γ

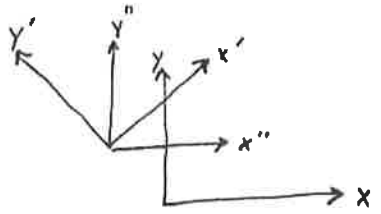
$$\Gamma = \frac{-2\pi R V e^{-i(\alpha+\beta)} + \frac{VR^2}{R} R e^{-i(\alpha+\beta)} e^{2i(\alpha+\beta)} 2\pi}{i} = 2 \cdot 2\pi R V \underbrace{\left(\frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i}\right)}_{\text{Remember this?}}$$

$$\boxed{\Gamma = 4\pi R V \sin(\alpha + \beta)}$$

Transform forces and Moments to $J(z)$ frame.

$$J(z) = z + \frac{c_1^2}{z}$$

From geometry



$$z' = r'e^{i\theta'}$$

$$z'' = z'e^{i\alpha}$$

$$z = z'' - me^{i\delta} = \underline{z'e^{\alpha} - me^{-i\delta}} = r'e^{i\theta'}e^{-} - me^{-i\delta}$$

solve for z'

$$z'e^{\alpha} = z + me^{i\delta} \Rightarrow z' = ze^{-i\alpha} + me^{-i\delta}e^{-i\alpha} = e^{-i\alpha}\left(z + me^{i\delta}\right)$$

Thus, $\frac{dz'}{dz} = e^{-i\alpha}$

~~And~~

Calculate $\frac{dw}{dJ}$ ~~to~~ to determine velocity magnitude

$$\frac{dw}{dJ} = \frac{dw}{dz'}\frac{dz'}{dz}\frac{dz}{dJ}$$

↖ we know this from cylinder

↖ offset and rotation map

↖ Joukowski map

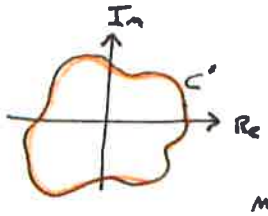
$$\frac{dw}{dJ} = \underbrace{\left(-V + \left(\frac{R^2}{z'^2}\right) - \frac{i\Gamma}{2\pi z'}\right)}\underbrace{\left(e^{-i\alpha}\right)}\underbrace{\left(1 - \frac{c_1^2}{z^2}\right)^{-1}}$$

$$= \left(-V + \frac{R^2}{e^{-2i\alpha}\left(z + me^{-i\delta}\right)^2} - \frac{i\,4\pi RV\sin(\alpha+\beta)}{e^{-i\alpha}\left(z + me^{i\delta}\right)}\right)\left(e^{-i\alpha}\right)\left(1 - \frac{c_1^2}{z^2}\right)^{-1}$$

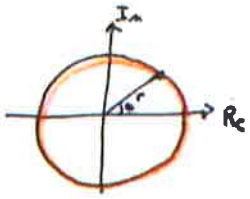
This gets messy....

Cauchy's 2nd Theorem

$$f(z) = \int_{C'} z^n \, dz$$



map to a circle



$$= \int_{C} z^n \, dz$$

$z = re^{i\theta}$

$dz = ire^{i\theta} d\theta$

- $n \neq -1$

$$= \int_{\theta=0}^{2\pi} (r^n e^{in\theta}) \, ire^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} ir^{n+1} e^{i(n+1)\theta} \, d\theta = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta = (ir^{n+1}) \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right|_0^{2\pi}$$

$$= ir^{n+1} \left(\frac{e^{i(n+1)2\pi} - 1}{i(n+1)} \right)$$

$e^{i2\pi \cdot (n+1)}$ is just multiples of revolutions around the Re-Im axis $= 1$

- 0 unless $n = -1$

- $n = -1$

$$f(z) = \int_C z^n dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = \int_0^{2\pi} i \, d\theta = i2\pi$$

Summary

Given a complex "polynomial" and a curve that encloses the point where $z^{-1} \to \infty$ ("singular point") ✓

$$g(z) = a_n z^n + \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + \cdots$$

The only non zero term in $f(z) = \int_{C'} g(z) \, dz$ is the a_{-1} term!

$$\boxed{f(z) = \int (a_n z^n + a_{-1} z^{-1} + a_0 z^0 + \cdots) \, dz = 2\pi i \, a_{-1}}$$

The force equation is

$$X - iY = \tfrac{1}{2}\rho i \oint \left(\frac{dw}{d\zeta}\right)^2 \frac{d\zeta}{dz}\, dz$$

The moment eqn is

$$M = -\tfrac{1}{2}\rho\, \mathrm{Real}\left(\int_c \left(\frac{dw}{d\zeta}\right)^2 \zeta\, d\zeta \right)$$

From the Cauchy 2nd theorem, we only need to track the $\frac{1}{z}$ terms!!

The Joukowski airfoil involves some algebra and substitution. So, let's focus on the fundamental with the untransformed, non-translated, non-rotated cylinder with a vortex.

$$w = -V\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi} \ln \frac{z}{R}$$

$$\frac{dw}{dz} = -V\left(1 + -\frac{R^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}$$

$$\left(\frac{dw}{dz}\right)^2 = V^2 \underbrace{\left(1 - \frac{R^2}{z^2}\right)^2}_{1 - 2\frac{R^2}{z^2} + \frac{R^4}{z^4}} + \frac{2Vi\Gamma}{2\pi z} + \frac{i^2\Gamma^2}{(2\pi)^2 z^2}$$

$$X - iY = \tfrac{1}{2}\rho i \oint \left(\frac{dw}{dz}\right)^2 dz$$

$$= \tfrac{1}{2}\rho i \oint \left(V^2 - 2V^2\frac{R^2}{z^2} + V^2\frac{R^4}{z^4} + \frac{2Vi\Gamma}{2\pi z} + \frac{i^2\Gamma^2}{(2\pi)^2 z^2} \right) dz$$

From the Cauchy theorem, only the $\frac{1}{z}$ terms are non zero.

$$= \tfrac{1}{2}\rho i \int \underbrace{\frac{2Vi\Gamma}{2\pi z}}_{g_1 = \frac{2Vi\Gamma}{2\pi}} dz + \underbrace{(\ldots)}_{0}$$

$$= \tfrac{1}{2}\rho i\,(2\pi i)\left(\frac{2Vi\Gamma}{2\pi}\right) = -i\rho V\Gamma$$

$$X - iY = -i\rho V\Gamma \Rightarrow \boxed{\begin{aligned} X &= 0 \\ Y &= \rho V \Gamma \end{aligned}}$$

The lift is $\rho V\Gamma$
The drag is zero

Returning to Joukowski: (we left the algebra elves to work ....)

Integrating $X - iY = \frac{1}{2}\rho i \int \left(\frac{dw}{dz}\right)^2 \frac{d\bar{f}}{dz}\, dz$ ← very messy even when only considering the ½ terms.

Gives

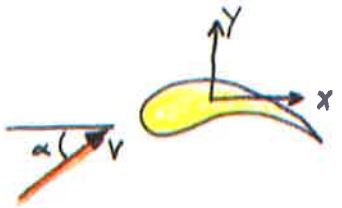
$$X - iY = \frac{1}{2}\rho i (2\pi i)\, i\frac{V\Gamma e^{i\alpha}}{\pi}$$

$$= -i\rho V\Gamma e^{i\alpha}$$

$$= -i\rho V\Gamma (\cos\alpha + i\sin\alpha)$$

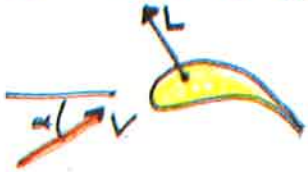
$$X = \rho V\Gamma \sin\alpha$$

$$Y = \rho V\Gamma \cos\alpha$$



But remember that X and Y are in the body frame, not the freestream frame

X and Y are components $(\sin\alpha, \cos\alpha)$ of a vector perpendicular to V. This is the lift vector. Zero drag.



$$L = \rho V\Gamma$$

Remember that we found Γ as a function of the geometry necessary for $V=0$ at TE.

$$\Gamma = 4\pi R V \sin(\alpha+\beta)$$

Substitute

$$L = \rho V\, 4\pi R V \sin(\alpha+\beta) = \underbrace{\tfrac{1}{2}\rho V^2}_{q} \cdot \underbrace{4R}_{\text{related to chord}} \cdot \underbrace{2\pi}_{\text{constant}} \cdot \underbrace{\sin(\alpha+\beta)}_{\text{angle}}$$

q = dynamic pressure

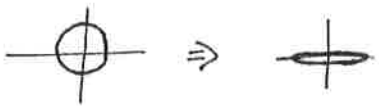
angle $\approx \alpha$ for small $\alpha+\beta$

Likewise, the moment at the center of the circle gives:

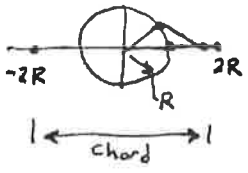
$$M_{cc} = \tfrac{1}{2}\rho V^2 \cdot 4\pi\epsilon^2 \cdot \sin 2\alpha$$

Flat plate airfoil

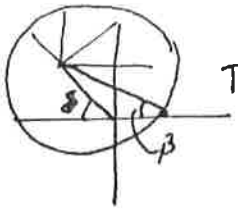
This is the transform with $C_1^2 = R^2$ and no circle offset.



The airfoil has a chord of $4R$ since $J = z + \frac{1}{z} = Re^{i\theta} + \frac{R^2}{R^2}e^{-i\theta} = R(e^{i\theta} + e^{-i\theta})$



Als



The circle is not offset thus $\beta = 0$

Lift

$$L = \tfrac{1}{2}\rho V^2 \cdot 4R \cdot 2\pi \cdot \sin(\alpha+\beta)$$

$$C_\ell = \frac{L}{q\bar{c}} = \frac{\tfrac{1}{2}\rho V^2 \cdot 4R \cdot 2\pi \cdot \sin(\alpha)}{\tfrac{1}{2}\rho V^2 \, 4R} \approx 2\pi\alpha - 2\pi\frac{\alpha^3}{6} + 2\pi\frac{\alpha^5}{120}$$

$2\pi\alpha$ is an approximation!

Good to 1% at 14° ← stall
2% at 20°
10% at 45°

Moment at midchord and ¼ chord

$$M_{1/2} = \tfrac{1}{2}\rho V^2 \, 4\pi C_1^2 \sin(2\alpha)$$

$$C_{m_{1/2}} = \frac{M}{q\bar{c}^2} = \frac{\tfrac{1}{2}\rho V^2 \, 4\pi C_1^2 \sin(2\alpha)}{\tfrac{1}{2}\rho V^2 \, (4C_1)^2} = \frac{\pi}{4}\sin 2\alpha$$

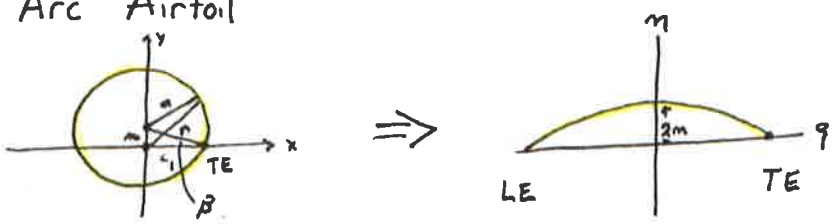
At ¼ chord,

$$C_{m_{1/4}} = C_{m_{1/2}} - \frac{C_\ell}{4} = \frac{\pi}{4}\sin(2\alpha) - \frac{2\pi\sin(\alpha)}{4}$$

$$= \frac{\pi}{2}\sin\alpha(\cos\alpha - 1)$$

≈ 0 for low values of α

Circular Arc Airfoil



$$x^2 + (y-m)^2 = a^2$$

and

$$c_1^2 + m^2 = a^2 \quad \Rightarrow \quad \eta = 2m \sin^2\theta$$

Define $f \equiv 2m$, thus camber as a percent of chord is $\frac{f}{c} = \frac{2m}{4c_1}$

Lift:

$$C_\ell = 2\pi \sin\left(\alpha + \operatorname{atan}\frac{m}{c_1}\right) \approx \boxed{C_\ell \approx 2\pi\left(\alpha + \frac{2f}{c}\right)}$$

$$= 2\pi \sin\left(\alpha + \operatorname{atan}\left(\frac{4m}{c}\frac{f}{2m}\right)\right)$$

$$= 2\pi \sin\left(\alpha + \operatorname{atan}\frac{2f}{c}\right)$$

Adding camber increases lift. A cambered airfoil has a zero lift angle of $-\frac{2f}{c}$

Moment:

$$C_{m_{1/4}} \approx -\frac{\pi f}{c}$$

Adding camber introduces a negative moment.

$$= -\frac{\pi}{2}\operatorname{atan}\frac{2f}{c}$$

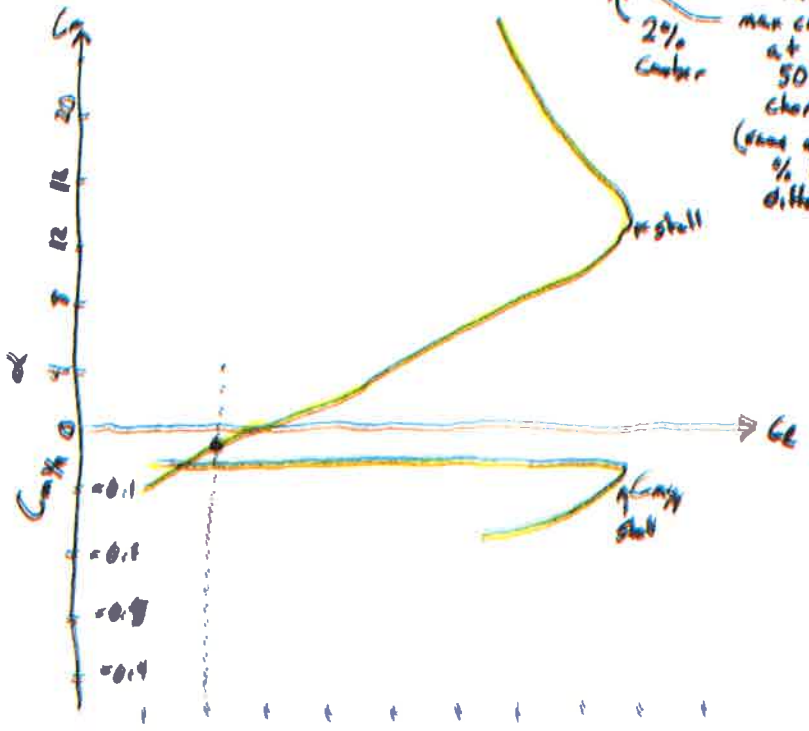
Compare to experiments

Pick an NACA 2509

2% camber

thickness

max camber at 50% chord (same as J(s)) % but different shape



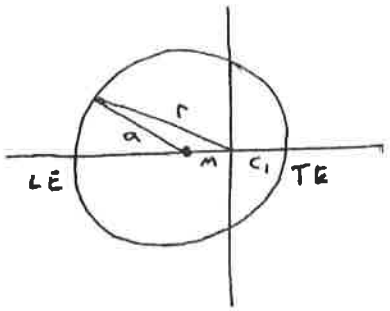
$$\alpha_{zL} \approx -\frac{2f}{c} = -0.04 \text{ rad}$$

$$\approx -2.3^\circ \checkmark$$

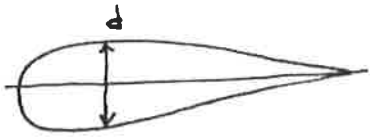
$$C_{m_{1/4}} \approx -\frac{\pi f}{c} = -\pi \cdot 0.02$$

$$\approx -0.063 \checkmark$$

Symmetrical Airfoil



$\Rightarrow J(z)$



Define $\epsilon \equiv \frac{m}{c_1}$ then $a = c_1(1+\epsilon) = c_1 + m$ and $r = a + m$

$= c_1(1+\epsilon) + m$

$= c_1(1+\epsilon) + c_1\epsilon$

$= c_1(1+2\epsilon)$

What is the chord?

Nose: $\zeta = z + \frac{c_1^2}{z} = re^{i\theta} + \frac{c_1^2}{r}e^{-i\theta} =$

$= -c_1(1+2\epsilon) - \frac{c_1^2}{c_1(1+2\epsilon)} \approx -2c_1(1+2\epsilon^2)$

Trailing Edge:

$\zeta = z + \frac{c_1^2}{z} = \cdots \approx 2c_1(1+2\epsilon^2)$

Chord $\approx 4c_1(1+2\epsilon^2)$

What is the thickness? What is the ϵ term?

$d = 3\sqrt{3}\,m$ (from some math) $\Rightarrow \frac{d}{c} = \frac{3\sqrt{3}\,c_1\epsilon}{4c_1(1+2\epsilon^2)} \approx \frac{3\sqrt{3}}{4}\epsilon$

$$\boxed{\epsilon \approx \frac{4}{3\sqrt{3}}\frac{d}{c}} \approx 77\%\,\frac{d}{c}$$

Max thickness is near 25% chord

Now for the stunning part....

Sym Airfoil (cont)

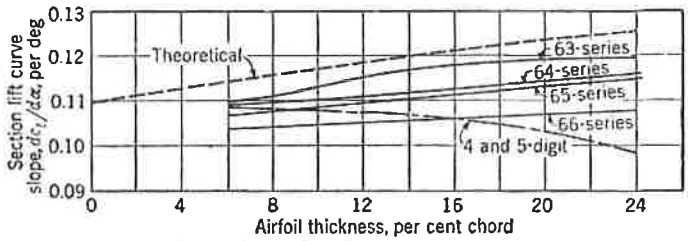
$$L = \tfrac{1}{2}\rho V^2\, 4a\, 2\pi \sin(\alpha + \cancel{\beta}^{\,0\ \text{sym.}})$$

$$C_\ell = \frac{L}{(\frac{1}{2}\rho V^2 c)} = \frac{\frac{1}{2}\rho V^2\, 4 c_1 (1+\epsilon)\, 2\pi \sin\alpha}{\frac{1}{2}\rho V^2\, 4 c_1 (1+2\epsilon^2)}$$

$$\boxed{C_\ell \approx 2\pi \frac{(1+\epsilon)}{(1+\epsilon^2)}\alpha} \Rightarrow C_{\ell_\alpha} \approx 2\pi \frac{1+\epsilon}{1+\epsilon^2} \quad \text{where } \epsilon \approx 77\%\ \tfrac{d}{c}$$

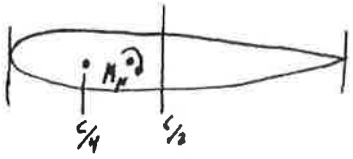
The slope of the lift curve increases with thickness!!

Compare with experiments



Moment:

You might guess that $C_{m_{c/4}} = 0$. Consider the following:



The transformation does not preserve lengths from the circle to the airfoil. The moment derived at the circle center is not the half chord.

$$M_{c/2} = \tfrac{1}{2}\rho V^2\, 4\pi c_1^2 \sin(2\alpha)$$

but

N is $c - (2+\epsilon)c_1$ from the LE

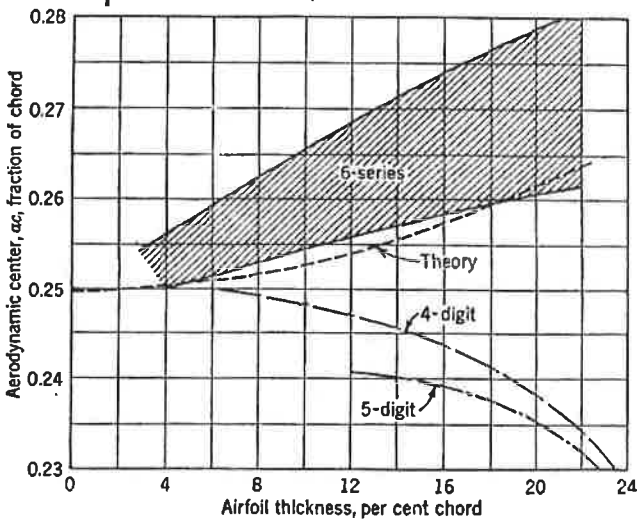
$$M_{a_c} = M_{c/2} - L\left(c - (2+\epsilon)c_1 - ac_{true}\cdot c\right)$$

Some math ... $\frac{dM}{d\alpha_0} = 0$

$$ac_{true} \approx \frac{1}{4} + \frac{\epsilon^2}{2}$$

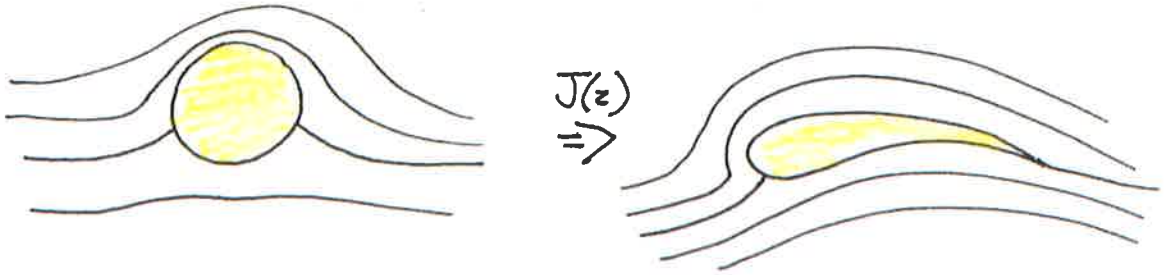
Adding thickness moves the true aerodynamic center aft.*

Comparison to experiments



Cylinder with Circulation

Now, we have the proper motivation to re-study the ~~circle~~ circle - potential flow.



Remember from fluids,

$$\psi = +V\left(r - \frac{R^2}{r}\right)\sin\theta$$

$$= \underbrace{+Vr\sin\theta}_{\text{Uniform flow}} \underbrace{+ -V\frac{R^2}{r}\sin\theta}_{\text{Doublet}}$$

Adding a vortex at the center of the circle gives

$$\psi = Vr\sin\theta - V\frac{R^2}{r}\sin\theta + \underbrace{\frac{\Gamma}{2\pi r}}_{\text{vortex}}$$

Low Γ, 2 stagnation pts.

Remember that adding a vortex is acceptable since the domain we want can be isolated from the vortex with a branch cut.

High Γ, 1 stagnation pt
($\Gamma = 4\pi V_\infty$)

Super High Γ, no stagnation pt on body.

So, which Γ do we pick?

Potential Function

- Take velocity potential and stream function and combine into a potential function

$$w = \phi + i\psi$$

warning!! w is not $\nabla \times V$ or V_z

For our cylinder (uniform + doublet + vortex)$_L$

$$w = -V\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi} \ln \frac{z}{R}$$

thus $\phi = \text{Real}(w)$
$\psi = \text{Imag}(w)$

- Write the velocity as a complex #

$$q = u + iv \quad \text{and} \quad \bar{q} = u - iv$$

$$|q| = |u+iv| = |u-iv| = \sqrt{u^2+v^2}$$

From definition of ϕ and ψ,

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Thus

$$|q| = |u+iv| = \left| \frac{\partial \phi}{\partial x} + i(-1)\left(\frac{\partial \psi}{\partial x}\right) \right| \text{ or}$$

$$= |u-iv| = \left| \frac{\partial \phi}{\partial x} - i(-1)\frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial}{\partial x}(\phi + i\psi) \right|$$

$$= \left| \frac{\partial}{\partial x}(w) \right| = \boxed{\left|\frac{\partial w}{\partial x}\right| = |q|}$$

Also

$$|q| = \left|\frac{\partial w}{\partial x}\right| = \left|\frac{\partial w}{\partial z}\frac{dz}{dx}\right| \quad \text{but } z = x + iy \Rightarrow \frac{dz}{dx} = 1$$

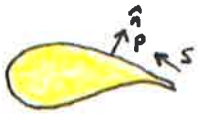
$$\boxed{|q| = \left|\frac{dw}{dz}\right|}$$

Lift and Moments in a potential flow

Given our mapping $\bigcirc \overset{J(\zeta)}{\Rightarrow} \frown$, we need the lift, drag and moment generated.

1) One method is to calculate pressures and surface normals, and integrate

$$\text{Force} = \oint_{\text{Airfoil}} p \cdot n \, dS$$



How can we find $\hat{n}$?

2) Apply momentum equation in a CV enclosing the airfoil. (FVA eqn 1.28)

$$\underbrace{\iiint \frac{\partial \rho v}{\partial t} dV}_{\text{steady state}} + \oiint \rho (V \cdot n) V \, dS = \iiint \rho f \, dV + \oiint -p\hat{n} \, dS + \underbrace{\oiint \bar{\bar{\tau}} \cdot \hat{n} \, dS}_{\text{No viscosity}} + \underbrace{B}_{\text{forces acting on airfoil}}$$

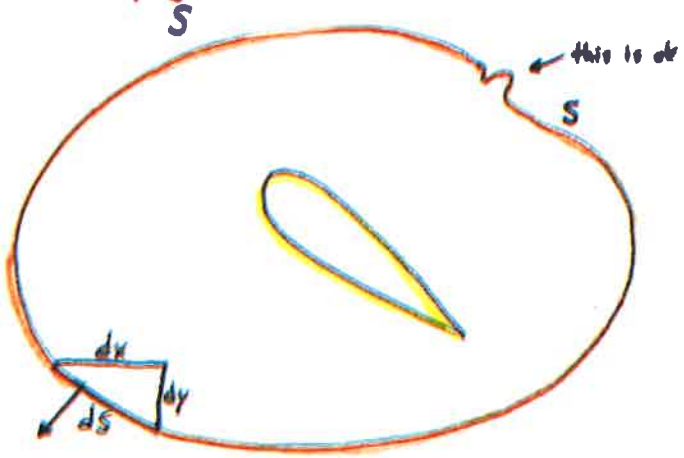
Simplify to

$$\int_S \rho (u n_x + v n_y + \cancel{w n_z}^{2D}) \begin{pmatrix} u \\ v \\ w \end{pmatrix} ds = -\int_S p \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}_{2D} dS + \underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{2D}}_{\text{force on airfoil} = +B}$$

Rearrange to solve for X, Y, Z

$$X = +\int_S p n_x \, dS + \rho \int_S (u n_x + v n_y) u \, dS$$

$$Y = +\int_S p n_y \, dS + \rho \int_S (u n_x + v n_y) v \, dS$$



$$\hat{n} = \frac{\begin{pmatrix} -dy \\ dx \end{pmatrix}}{ds}$$

L+M continued

$$X = -\int_S p\,dy + \rho\int_S (v\,dx - u\,dy)\,u$$

$$Y = \int_S p\,dx + \rho\int_S (v\,dx - u\,dy)\,v$$

Compute force in complex frame

$$X - iY = -\int_S p(dy + i\,dx) + \rho\int_S (u - iv)(v\,dx - u\,dy)$$

For low speeds (incomp)

$$p = p_0 - \tfrac{1}{2}\rho(u^2 + v^2)$$

switch signs from above

$$X - iY = -\int_S \left(p_0 - \tfrac{1}{2}\rho(u^2+v^2)\right)(dy + i\,dx) - \rho\int_S (u - iv)(u\,dy - v\,dx)$$

But, $\int_S p_0\,dS = 0$

Thus

$$\boxed{X - iY = \frac{1}{2}\rho i \int_S \left(\frac{dw}{dz}\right)^2 dz}$$

Remember, $|q| = \left|\frac{dw}{dz}\right|$ is the velocity

Moment:

Similar derivation

$$\boxed{M = -\frac{1}{2}\rho\,\mathrm{Real}\left(\int_C \left(\frac{dw}{dz}\right)^2 z\,dz\right)}$$

These are the Blasius equations developed by .... well .... Blasius in 1908.