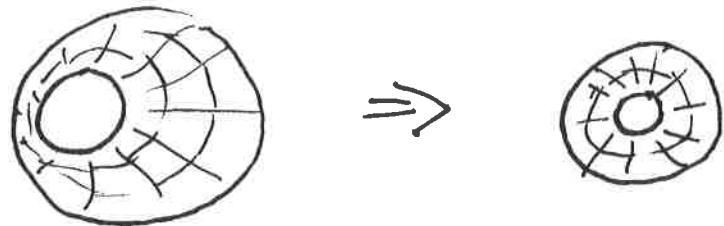
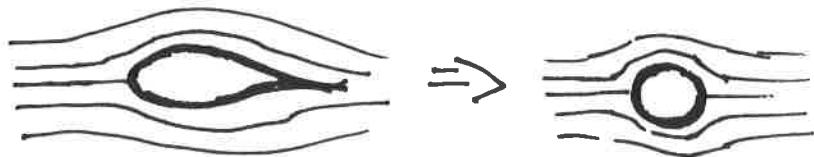
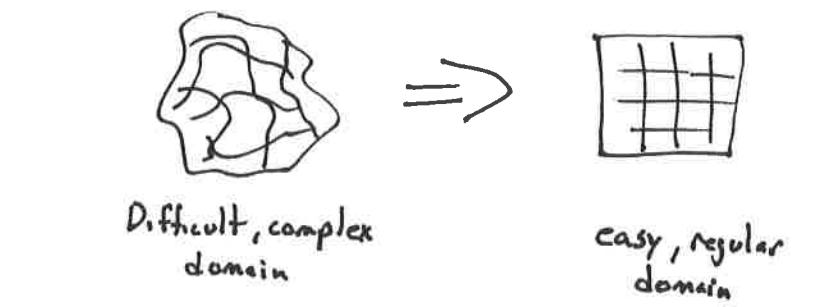


GES 554

Lesson 47

Conformal Mapping

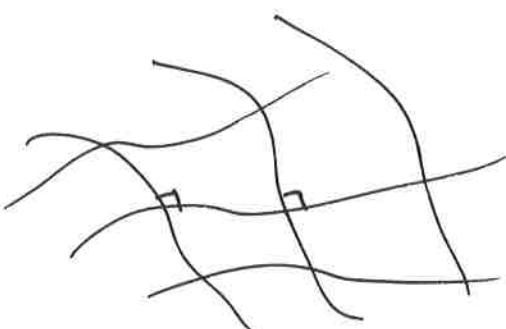
# Conformal Mapping



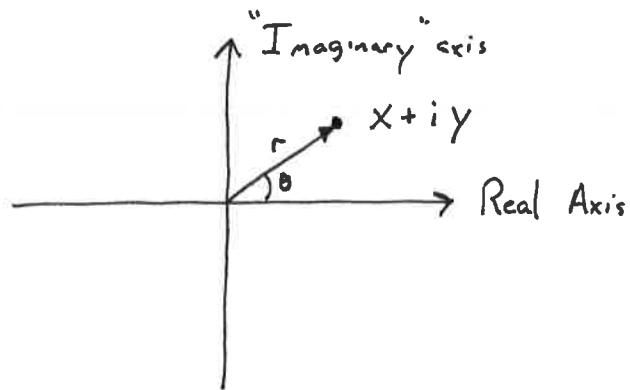
Apply a map from  $z$  to  $w$ :

$$w = f(z)$$

where  $z = x + iy$



# Complex #s



$$z = x + iy \\ = r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right)$$

Ex:

$$z = \sqrt{2} e^{i\pi/4}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

~~$$|z| = \sqrt{z} = \sqrt{x^2 + y^2}$$~~

$$x = \sqrt{2} \cos \frac{\pi}{4} = 1$$

$$y = \sqrt{2} \sin \frac{\pi}{4} = 1$$

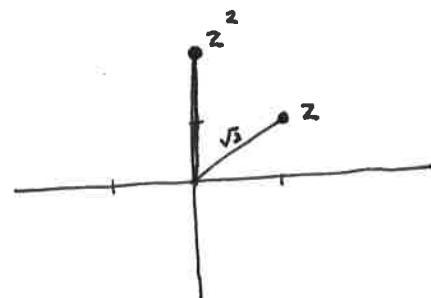
$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$$

Ex:

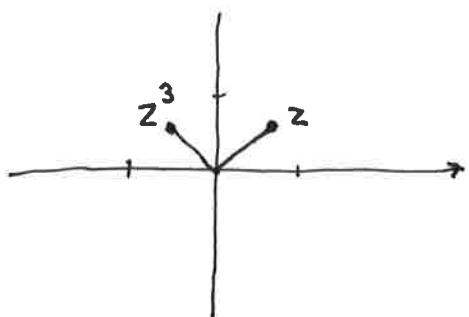
$$z = \sqrt{2} e^{i\pi/4} \quad w = z^2$$

$$w = z^2 = \sqrt{2}^2 \left(e^{i\pi/4}\right)^2 = 2 e^{i\pi/2}$$

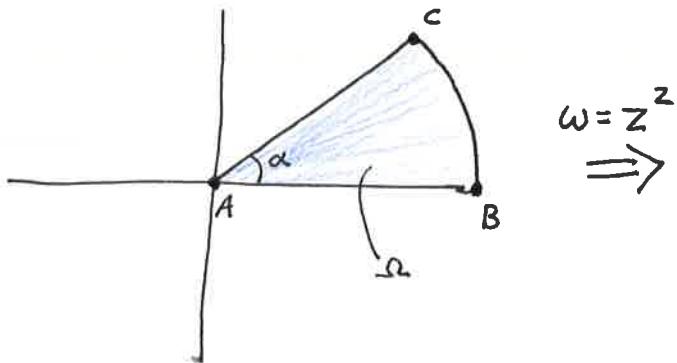


$$Ex: \quad z = \cancel{\frac{1}{\sqrt{2}}} e^{i\pi/4} \quad w = z^3$$

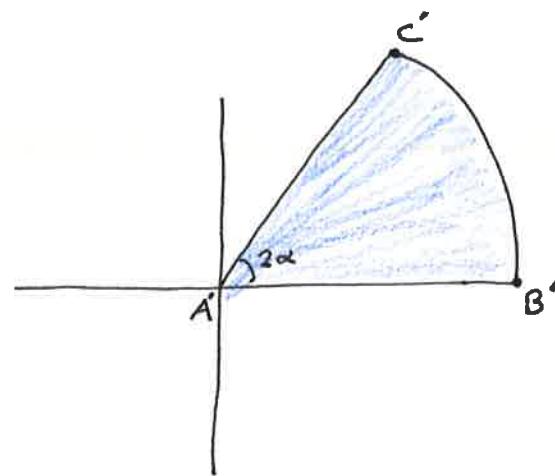
$$w = \cancel{\sqrt{2}} e^{i3\pi/4}$$



Mapping  $z^2$



$$z = x + iy \Rightarrow$$



$$\begin{aligned} w &= z^2 = (x+iy)^2 \\ &= x^2 - y^2 + i(2xy) \end{aligned}$$

$$z = r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$w = z^2 = r^2 e^{i2\theta}$$

$$r = |z| = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$$

$$= x^2 + y^2$$

$$A' = A^2 = (0+i0)^2 = 0$$

$$\theta = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$$

$$B' = B^2 = (b+i0)^2 = b^2 + i0$$

$$C' = C^2 = (c_r + ic_i)^2 = (c_r^2 - c_i^2) + i(2c_r c_i)$$

# Analytic functions

$$f(z) = f(x+iy) \Rightarrow z = x + iy$$

derivatives

$$\frac{df}{dx} = \frac{df}{dz} \frac{dz}{dx}^i = \frac{df}{dz}$$

$$\frac{df}{dy} = \frac{df}{dz} \frac{dz}{dy}^i = i \frac{df}{dz}$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dz}\left(\frac{df}{dx}\right) \frac{dz}{dx}^i = \frac{d}{dz}\left(\frac{df}{dz}\right) = \frac{d^2f}{dz^2}$$

$$\frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{dz}\left(\frac{df}{dy}\right) \frac{dz}{dy}^i = \frac{d}{dz}\left(i \frac{df}{dz}\right)i = i^2 \frac{df}{dz^2} = -\frac{d^2f}{dz^2}$$

Laplace's Eqn.

$$\nabla^2 u = 0 = \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} =$$

substitute above

$$0 = \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0 \quad \checkmark$$

$$z = x + iy \text{ satisfies } \nabla^2 u = 0$$

In fact, any  $z = (x+iy)^n$  satisfies Laplace's eqn.

We call any  $f(z)$  that can be written as  $(x+iy)^n$  and converges an Analytic function.

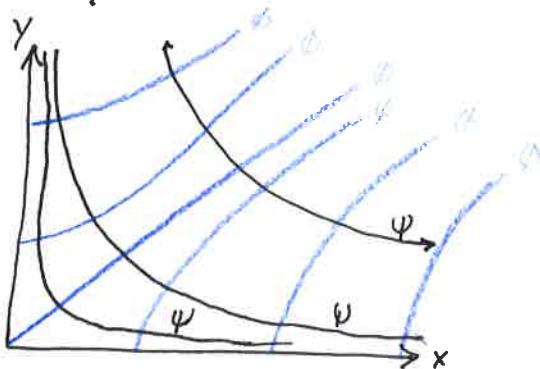
# Cauchy - Riemann

If a function  $f(x+iy)$  converts into  $u(x,y) + i s(x,y)$ ,

the function is analytic if

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}}$$

For example, 90° corner flow



$$\psi(x,y) = Ax y$$

$$\phi(x,y) = -\frac{1}{2}A(x^2 - y^2)$$

My aero book skipped the minus!

This should (and does) satisfy  $\nabla^2 \psi = 0$  and  $\nabla^2 \phi = 0$

Use C-R to verify analytic...

$$f(z) = f(x+iy) = \overbrace{\psi(x,y)}^{u(x,y)} + i \overbrace{\phi(x,y)}^{s(x,y)}$$

$$= Axy + i \frac{1}{2}A(x^2 - y^2)$$

Test

$$\frac{\partial u}{\partial x} \stackrel{?}{=} \frac{\partial s}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial \psi}{\partial x} = Ay \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{\partial \phi}{\partial y} = -\frac{1}{2}A(-2y)$$

and  $Ay = Ay \quad \checkmark$

$$\frac{\partial u}{\partial y} \stackrel{?}{=} -\frac{\partial s}{\partial x} \Rightarrow Ax = -\left(\frac{1}{2}\right)A2x \quad \checkmark$$

Warning!! Not all  
aero textbook provide  
consistent  $\psi$  and  $\phi$ !  
Ex:  $\phi_{\text{aero}} = -\phi_{\text{math}}$

L47 p4

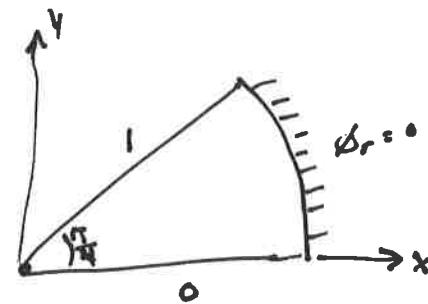
$$\nabla^2 \phi = 0$$

$$\phi(r, 0) = 0$$

$$\phi(r, \frac{\pi}{4}) = 1$$

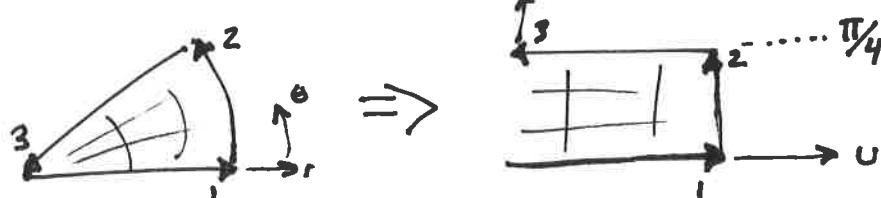
$$\phi_r(1, \theta) = 0$$

$$0 < r < 1$$

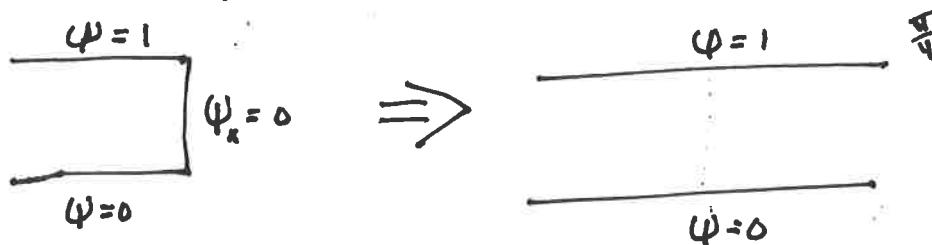


$$\text{Map } w = \log|z| + i \arg(z) = \log(z)$$

$$= \log|x+iy| + i \arg(x+iy) = u + iv$$



Find solution to  $\nabla^2 \psi$  in the new domain



$$\psi(u, v) = \frac{4}{\pi} v \quad \text{with } v = \arg(x+iy) = \theta$$

Convert back to  $\phi$

- By inspection ~~we notice~~ we notice that  $\psi$  is only a function of  $v$ .

$$\phi = \frac{4}{\pi} \theta$$

- solve for  $r, \theta$  in terms of  $u, v$ .

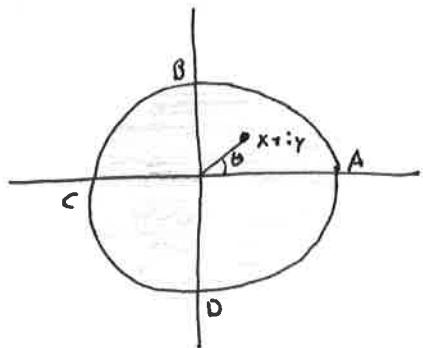
$$\begin{aligned} u + iv &= \log|z| + i \arg(z) \\ &= \log|z| + i\theta \quad \Rightarrow \quad v = \theta \\ &= \log(r) + i\theta \end{aligned}$$

$$u = \log(r)$$

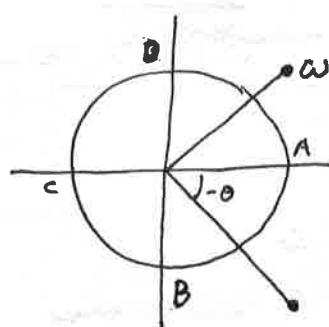
max

$$v = \theta$$

Mapping  $\frac{1}{z}$



$$w = \frac{1}{z} \Rightarrow$$



$$z = x + iy$$

$$= r \cos \theta + i r \sin \theta$$

$$w = \frac{1}{z} = \frac{1}{x+iy}$$

multiply + divide by conjugate

$$w = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x}{r^2} - i \frac{y}{r^2} \quad \text{mixed coordinate system}$$

$$= \frac{\cos \theta}{r} - i \frac{\sin \theta}{r}$$

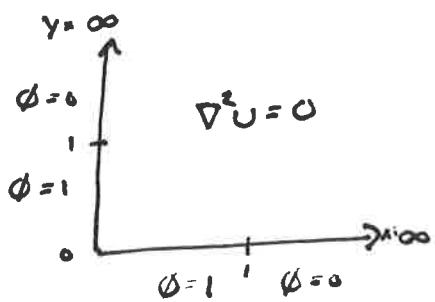
$$z = r e^{i\theta}$$

~~cancel r~~

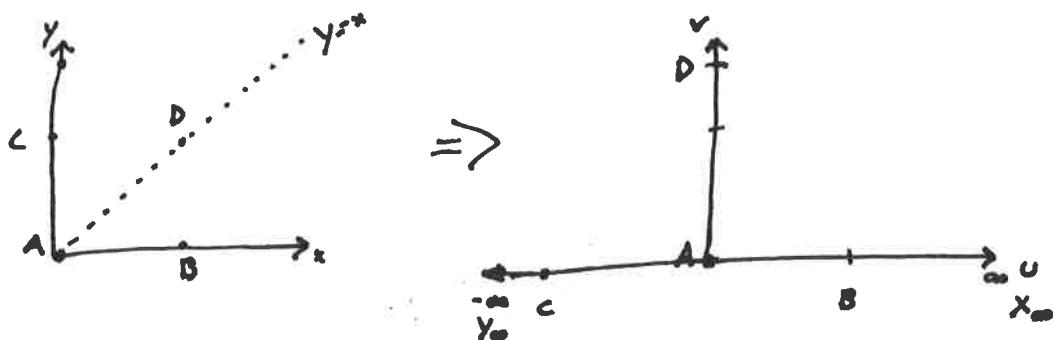
$$w = \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

Ex:

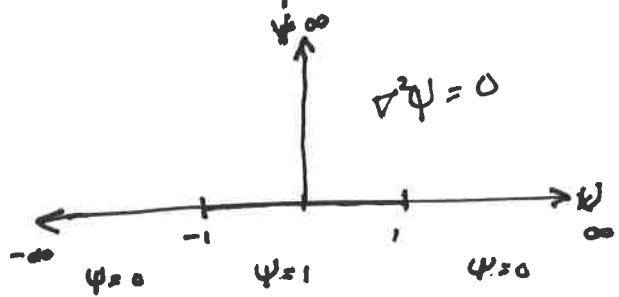
L 47 P3



$$\text{Map } w = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i2xy$$

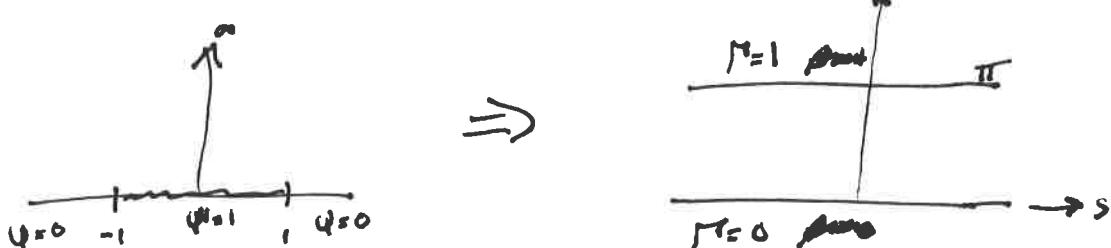


Solve new problem



$\psi(z) =$  This is a tough problem.  
I tried and failed with Fourier transforms.....

Map to a 3rd problem



$$w = \log\left(\frac{z-1}{z+i}\right)$$

Solution is  $\Gamma(s,t) = \frac{t}{\pi}$

$$w = \log \left| \frac{z-1}{z+1} \right| + i \arg \left( \frac{z-1}{z+1} \right) = s + it$$

We know that  $\Gamma$  is only a function of  $t$ .

$$\begin{aligned} t &= \arg \left( \frac{z-1}{z+1} \right) = \arg \left( \frac{u+iv-1}{u+iv+1} \right) \\ &= \arg \left( \frac{u^2+v^2-1+2iv}{u^2+v^2+2u+1} \right) \quad \text{some algebra....} \\ &= \tan^{-1} \left( \frac{2v}{u^2+v^2-1} \right) \quad \text{since } \arg = \tan^{-1} \left( \frac{v}{u} \right) \end{aligned}$$

Complete solution

$$\Gamma = \frac{t}{\pi} = \frac{1}{\pi} \tan^{-1} \left( \frac{2v}{u^2+v^2-1} \right) = \psi$$

now subst  $u$  and  $v$ .  $u = x^2 - y^2$  and  $v = 2xy$

~~Don't use~~

$$\phi = \frac{1}{\pi} \tan^{-1} \left( \frac{4xy}{(x^2-y^2)^2 + 4x^2y^2 - 1} \right)$$

This is nothing short of amazing.

No Fourier transforms. No integrals.

2 Maps and some algebra.

$$A + iB$$

$$= \frac{v+i\sqrt{-1}}{v+i\sqrt{+1}}$$

$$= \frac{(v-1) + i\sqrt{}}{(v+1) + i\sqrt{}}$$

$$= \frac{(v-1) + i\sqrt{}((v+1) - i\sqrt{})}{(v+1)^2 + \sqrt{^2}}$$

$$= \frac{(v-1)(v+1) - i\sqrt{(v-1)} + i\sqrt{(v+1)} + \sqrt{^2}}{(v+1)^2 + \sqrt{^2}}$$

$$= (v^2 - 1 + \sqrt{^2}) + i(2\sqrt{})$$

Congratulations

We finished the book