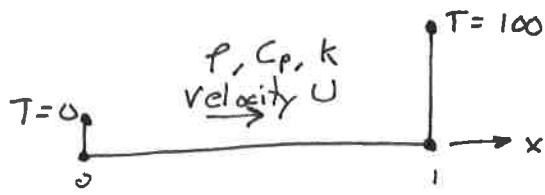


Advection-Diffusion

Given a steady state advection diffusion ODE, find the solution numerically.



Gov Egu:

$$\rho C_p \cdot u \cdot \frac{dT}{dx} - k \frac{d^2T}{dx^2} = 0$$

Analytical Solution

$$T(x) = \frac{100}{1-e^c} - \frac{100}{1-e^c} e^{cx} \quad \text{where } c = \frac{\rho C_p u}{k}$$

Q: How is molecular information transported in this situation?

A: pure diffusion ($U=0$) is Brownian motion.

- Information is equally mixed from neighbors

A: pure advection ($U \neq 0$, $K=0$)

- Information only moves along characteristic.
- At a stationary point at x , we want solution states from upwind locations

Our numerical solution must reflect physical behavior. (upwind for advection, central for diffusion)

Finite difference with central derivatives

$$\rho C_p U T_x - K T_{xx} = 0 \xrightarrow{\text{non-dim}} C T_x - T_{xx} = 0$$

where $C = \frac{\rho C_p U}{K}$

Central difference

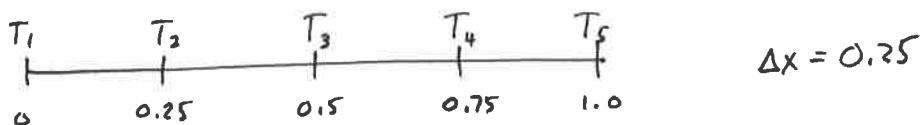
$$T_x \approx \frac{T_{i+1} - T_{i-1}}{2\Delta x}$$

$$T_{xx} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2}$$

$$C \frac{T_{i+1} - T_{i-1}}{2\Delta x} - \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} = 0$$

$$T_i = \frac{(T_{i+1} + T_{i-1})}{2} - C T_{i+1} \Delta x + C T_{i-1} \Delta x$$

$$= T_{i+1} \underbrace{\left(\frac{1}{2} - C \Delta x \right)}_{A_E} + T_{i-1} \underbrace{\left(\frac{1}{2} + C \Delta x \right)}_{A_W}$$



$$\begin{bmatrix} -A_W + 1 & -A_E & & & \\ -A_W & +1 & -A_E & & \\ -A_W & +1 & -A_E & & \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

⇒ $\begin{bmatrix} 1 & -A_E & & & \\ -A_W & 1 & -A_E & & \\ -A_W & -A_E & 1 & -A_E & \\ & & & 1 & -A_E \\ & & & -A_E & 1 \end{bmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = 0$

$+100 A_E$

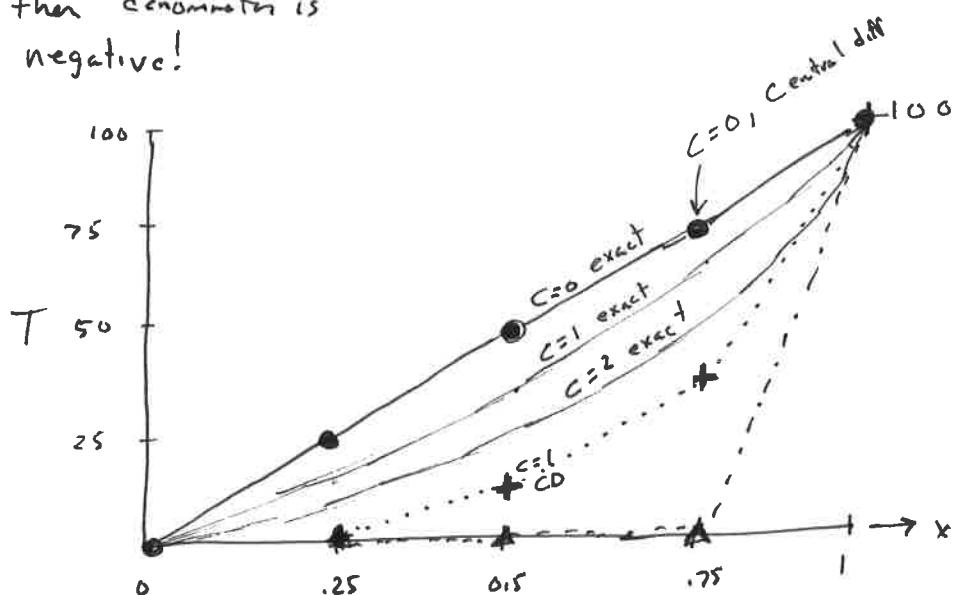
$$\begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix} = \frac{\begin{bmatrix} 1 - A_E A_w & A_E & A_E^2 \\ A_w & 1 & A_E \\ A_w^2 & A_w & 1 - A_E A_w \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ +100 A_E \end{pmatrix}}{1 - 2 A_E A_w}$$

if $A_E = 0$, no "diffusion" $A_E = \frac{1}{2} - \frac{C}{2} \alpha x$

$$T_2 = \frac{+100 A_E^3}{1 - 2 A_E A_w} \quad T_3 = \frac{+100 A_E^2}{1 - 2 A_E A_w} \quad T_4 = \frac{+100 A_E + 100 A_E^2 A_w}{1 - 2 A_E A_w}$$

if $A_E \cdot A_w > \frac{1}{2}$

then denominator is negative!



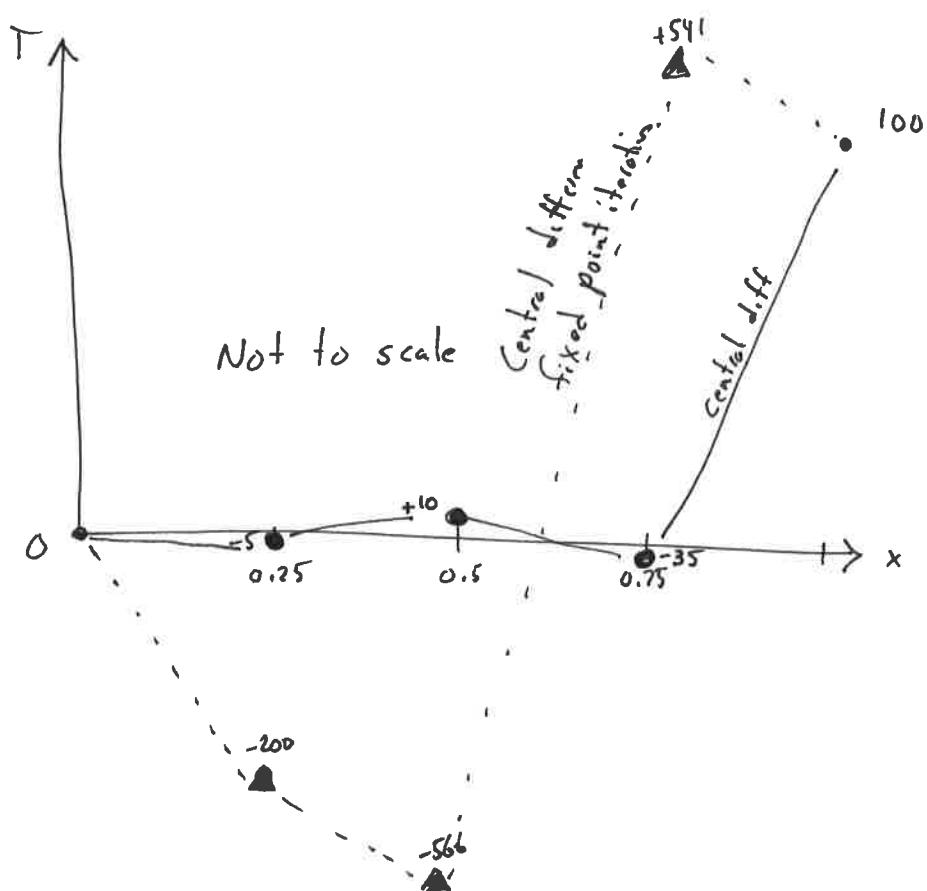
Finite difference with central difference derivatives
and fixed point iteration

$$T_2 = A_E T_3$$

$$T_3 = A_W T_2 + A_E T_4$$

$$T_4 = A_E 100 + A_W T_3$$

When C increases, the fixed point approximation becomes unstable
even when the central difference method remains stable (but wrong)!



Backwards differencing

$$T_x \approx \frac{T_{i-1} - T_i}{\Delta x} \approx \frac{T_i - T_{i-1}}{\Delta x}$$

$$T_{xx} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2}$$

what about reverse flow
 $U < 0$

$$T_x \approx \frac{T_{i+1} - T_i}{\Delta x} \approx \frac{T_{i+1} - T_i}{\Delta x} \quad \text{Not the same}$$

For advection, pick the derivative in the upwind direction.

~~$$C \frac{T_i - T_{i-1}}{\Delta x} - \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} = 0$$~~

$$\Delta x C T_i - \Delta x C T_{i-1} - T_{i-1} + 2T_i - T_{i+1} = 0$$

$$(2 + \Delta x C) T_i = \Delta x C T_{i-1} + T_{i-1} + T_{i+1}$$

$$T_i = \underbrace{\frac{(\Delta x C + 1)}{(\Delta x C + 2)} T_{i-1}}_{A_w} + \underbrace{\frac{1}{\Delta x C + 2} T_{i+1}}_{A_E}$$

A_w and A_E are never negative or zero.

Stable routine

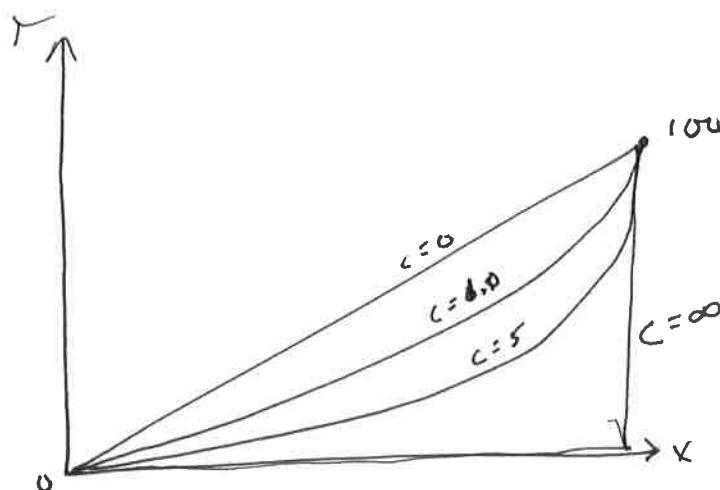
From before, the 5 point Finite Diff' approximation is

$$T_2 = \frac{100 A_E^3}{1 - 2A_E A_W}, \quad T_3 = \dots, \quad T_4 = \dots$$

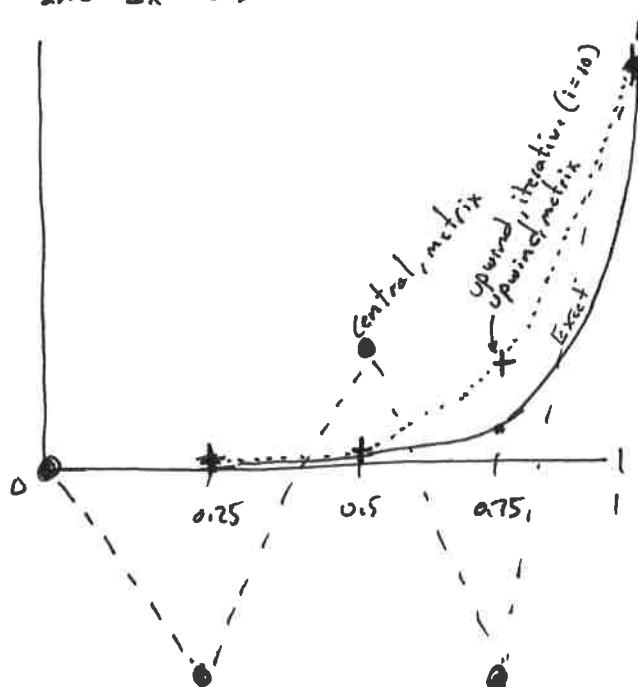
stable as $\Delta x C \rightarrow 0$ $T_2 = \frac{100 (\frac{1}{2})^3}{1 - 2 \cdot \frac{1}{4}} = \frac{100 \cdot \frac{1}{8}}{1 - \frac{1}{2}} = 25 \checkmark$

stable as $\Delta x C \rightarrow \infty$

$$T_2 = \frac{100 \left(\frac{1}{\Delta x C} \right)^3}{1 - 2 \left(\frac{\Delta x C}{\Delta x C} \cdot \frac{1}{\Delta x C} \right)} \Rightarrow 0 \checkmark$$



If: $C = 10$ and $\Delta x = 0.25$



- fixed point Central No solution, unstable, FAILS
- Matrix Central Fails but solution
- + Matrix Upwind OK, overpredicts

Upwinding represents
the physics of
advection