Biharmonic Equation

$$\nabla^4 U = 0 \qquad (\text{aka } \nabla^2 \nabla^2 U = 0)$$

- By inspection, every solution of $\nabla^2 U = 0$ also is a solution of $\nabla^4 U = 0$

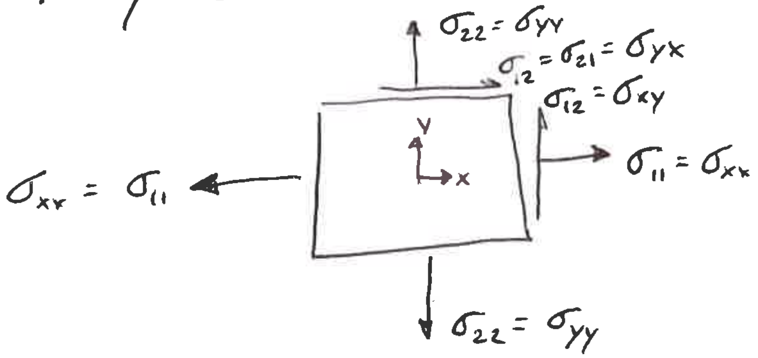
But not every solution of the biharmonic equation satisfies Laplace's equation.

- Cartesian coordinates

$$\nabla^4 U = \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0$$

Warning: Not every conformal map for $\nabla^2 U = 0$ works for $\nabla^4 U = 0$

- Airy Stress function ϕ, $\nabla^4 \phi = 0$



$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

$$\sigma_{xx} = \sigma_{11} = \phi_{yy}$$

$$\sigma_{yy} = \sigma_{22} = \phi_{xx}$$

$$\sigma_{xy} = \sigma_{12} = \sigma_{21} = -\phi_{xy}$$

Compatibility? $\nabla^2(\sigma_{11} + \sigma_{22}) = 0$ ✓

$$\nabla^2(\phi_{yy} + \phi_{xx}) = 0$$

$$\phi_{yyxx} + \phi_{yyyy} + \phi_{xxxx} + \phi_{xxyy} = 0$$

definition of $\nabla^4 \phi = 0$ ✓

- Any polynomial of order less than 4 is biharmonic. (trivial soln!)

$$1$$
$$x \qquad y$$
$$x^2 \qquad xy \qquad y^2$$
$$x^3 \qquad x^2 y \qquad xy^2 \qquad y^3$$
$$x^3 y \quad x^3y^2 \quad x^2y^2 \quad x^3y^3 \quad x^2y^3 \quad xy^3$$

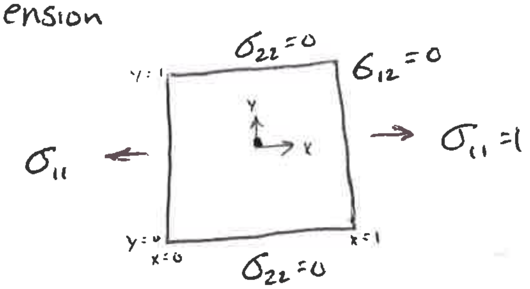
$$\phi = A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y + Ixy^2 + Jy^3 + Kx^3y + Lx^2y^2 + Mxy^3 + Nx^3y^3 + Px^3y^2 + Qx^2y^3$$

$$\sigma_{11} = \phi_{yy} = 2F + 2Ix + 6Jy + 2Lx^2 + 6Mxy + 6Nx^3y + 2Px^3 + 6Qx^2y$$

$$\sigma_{22} = \phi_{xx} = 2D + 6Gx + 2Hy + 6Kxy + 2Ly^2 + 6Nxy^3 + 6Pxy^2 + 2Qy^3$$

$$\sigma_{12} = E + 2Hx + 2Iy + \cdots + 3Kx^2y + 4Lxy + 3My^2 + 9Nx^2y^2 + 6x^2y + 6xy^2$$

- Tension



$$\sigma_{11} = \begin{cases} 1 & \text{on left + right faces} \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_{22} = 0$ on all surfaces BCs

$\sigma_{12} = 0$ on all " "

$$\sigma_{11}(x=0, y=0..1) = 1 = 2F + 2I \cdot 0 + \underbrace{6Jy}_{\text{no y dependence}}$$
$$\sigma_{11}(x=1, y=0..1) = 1 = 2F + 2I \cdot 1 + \cancel{6Jy}$$

$\Rightarrow$ $1 = 2F + 0$, $1 = 2F + 2I$

$F = \frac{1}{2}, \quad I = 0$

$$\sigma_{22}(y=0, x=0..1) = 0 = 2D + \underbrace{6Gx}_{\text{no x dep}} + 2H0$$
$$\sigma_{22}(y=1, x=0..1) = 0 = 2D + \cancel{6Gx} + 2H$$

$D = 0, \quad H = 0$

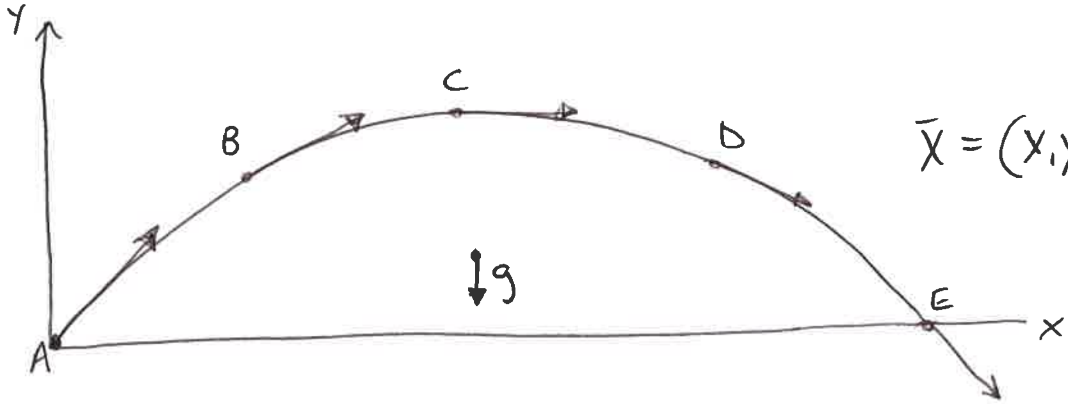
$$\sigma_{12}(\text{all}) = E + 2\overset{0}{\cancel{H}}x + 2\overset{0}{\cancel{I}}y = 0 \qquad E = 0$$

$$\boxed{\phi = \frac{1}{2}y^2}$$

$$\sigma_{11}(x,y) = \phi_{yy} = 1 \quad \checkmark$$

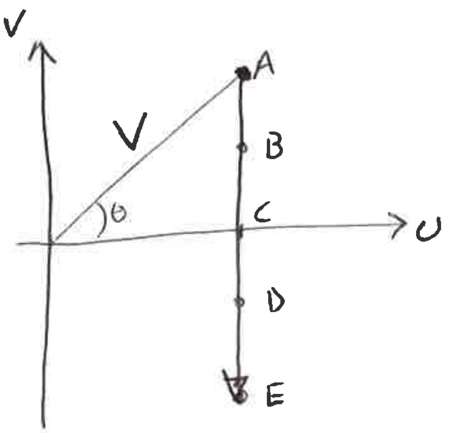
Hodograph

X-Y trajectory

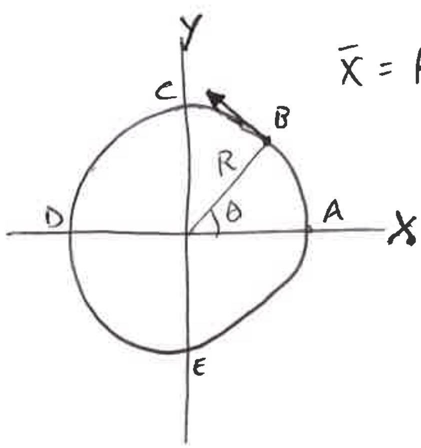


$\bar{X} = (x, y) = ?$

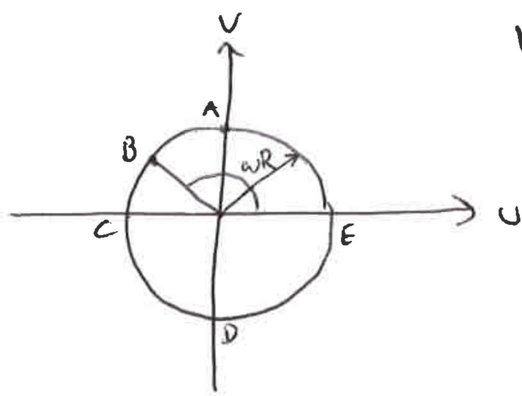
Hodograph



$V = (U_0 , V_0 - gt)$



$\bar{X} = Re^{i\omega t}$



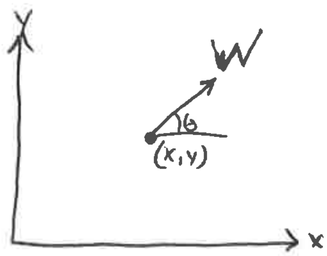
$\bar{V} = \frac{d\bar{x}}{dt} = i\omega Re^{i\omega t}$

Hodograph

"write" in Greek

Greek "path

Ref: Analytical Fluid Dynamics
George Emanuel



$$u = W\cos\theta$$
$$v = W\sin\theta$$

$$u = \phi_x \quad , \quad v = \phi_y \quad , \quad u = \frac{\rho_0}{\rho}\psi_y \quad , \quad v = -\frac{\rho_0}{\rho}\psi_x$$

- ϕ and ψ are functions of x and y $\quad \phi(x,y)$, $\psi(x,y)$

$$d\phi = \frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy = u\,dx + v\,dy = w\cos\theta\,dx + w\sin\theta\,dy$$

$$d\psi = \frac{d\psi}{dx}\,dx + \frac{d\psi}{dy}\,dy = -\frac{\rho}{\rho_0}v\,dx + \frac{\rho}{\rho_0}u\,dy = -\frac{\rho}{\rho_0}w\sin\theta\,dx + \frac{w\rho}{\rho_0}\cos\theta\,dy$$

- Solve for dx and dy

$$w\begin{bmatrix}\cos\theta & \sin\theta \\ -\frac{\rho}{\rho_0}\sin\theta & \frac{\rho}{\rho_0}\cos\theta\end{bmatrix}\begin{pmatrix}dx \\ dy\end{pmatrix} = \begin{pmatrix}d\phi \\ d\psi\end{pmatrix} \quad \Rightarrow \begin{pmatrix}dx \\ dy\end{pmatrix} = \frac{1}{w}\begin{bmatrix}\cdots\end{bmatrix}$$

seen this transform before?

After some work

$$(M^2-1)\,\psi_{\theta\theta} = (1+M^2)\,w\,\psi_w + w^2\psi_{ww}$$

or

$$\tau^2(1-\tau)\,\psi_{\tau\tau} + \tau\left(1 + \frac{2-\gamma}{\gamma-1}\tau\right)\psi_\tau + \frac{1}{4}\left(1 - \frac{\tau}{\tau^*}\right)\psi_{\theta\theta} = 0$$

where $M^2 = \left(\frac{2}{\gamma-1}\right)\left(\frac{\tau}{1-\tau}\right)$

When $M = 0$

$$-\psi_{\theta\theta} = w\psi_w + w^2\psi_{ww}$$

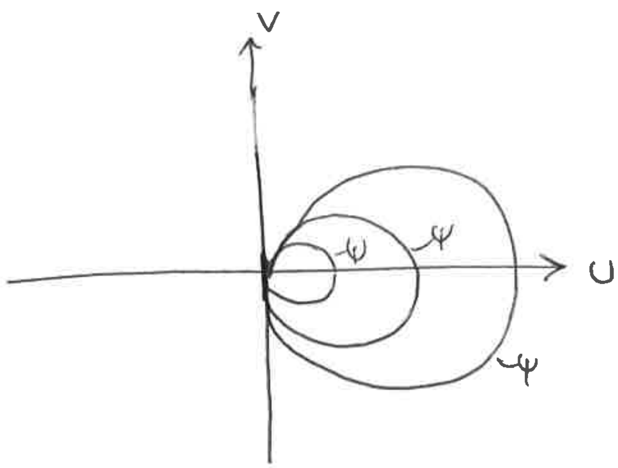
one solution is

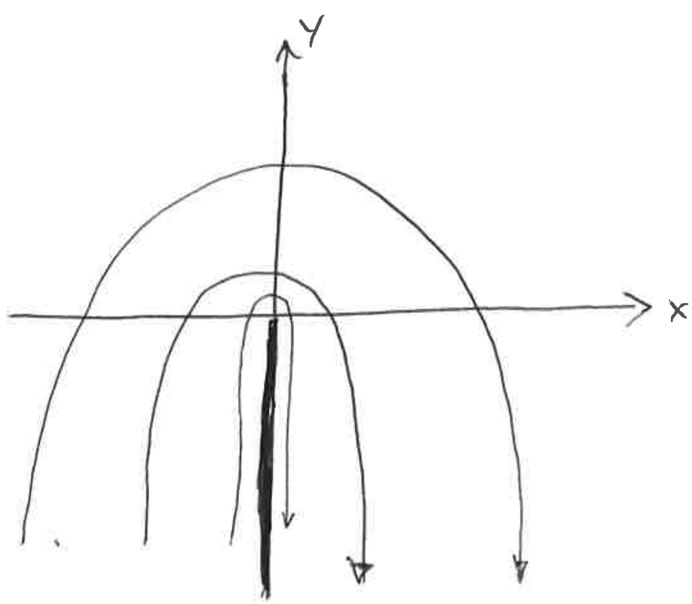
$$\psi = \frac{\cos\theta}{w}$$

$$U = w\cos\theta = w^2\psi$$

$$V = w\sin\theta = w(\sin^2\theta)^{1/2} = w(1-\cos^2\theta)^{1/2}$$

$$= w(1-\psi^2 w^2)^{1/2}$$





Compressible Solution (Ringleb in 1940)

$$\psi = \frac{\cos\theta}{\tau^{1/2}}$$

Advantages:

- Compressible Euler solution to fluid dynamics!
- Simple solution form in ψ
- perfect for verification + validation (V+V)

  verification: compare to exact soln

  validation: compare to experimental data

- Shows conceptual behavior of supersonic ↔ subsonic flows.

Disadvantages:

- Solution in velocity frame.
- Very unlikely to fit x, y boundary conditions for real flows!
- Needs a specially prepared domain and BCs for Verification of numerical solutions
- Other solutions? Good luck.

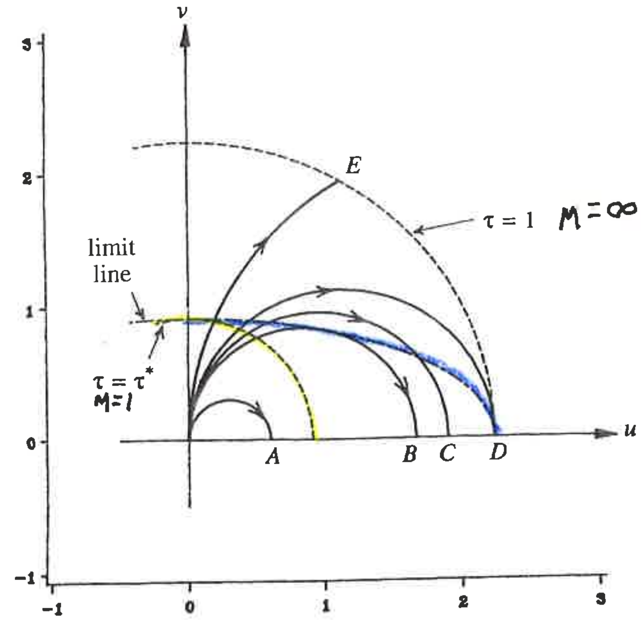


FIGURE 7.5 Compressible Ringleb solution in the hodograph plane, $\gamma = 7/5$.

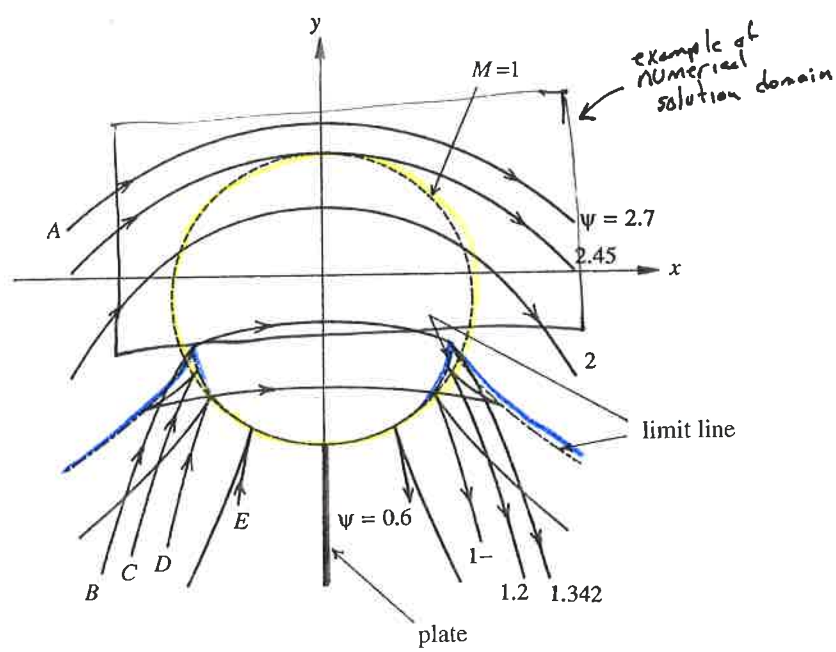


FIGURE 7.6 Streamlines, sonic line, and limit line of the compressible Ringleb solution, $\gamma = 7/5$.