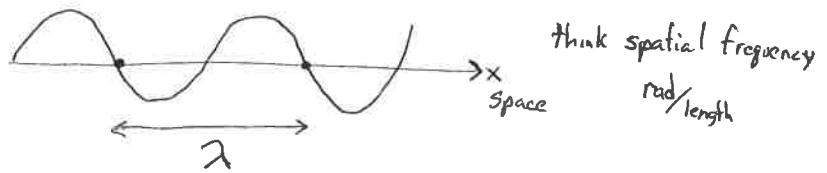


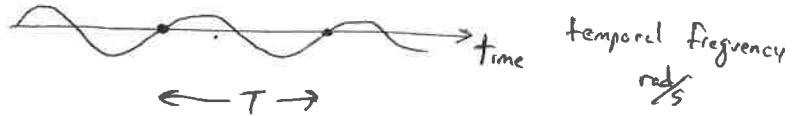
Waves

$$\text{wavenumber } K = \frac{2\pi}{\lambda}$$



think spatial frequency
rad/length

$$\text{frequency } \omega = \frac{2\pi}{T}$$



temporal frequency
rad/s

Traveling wave

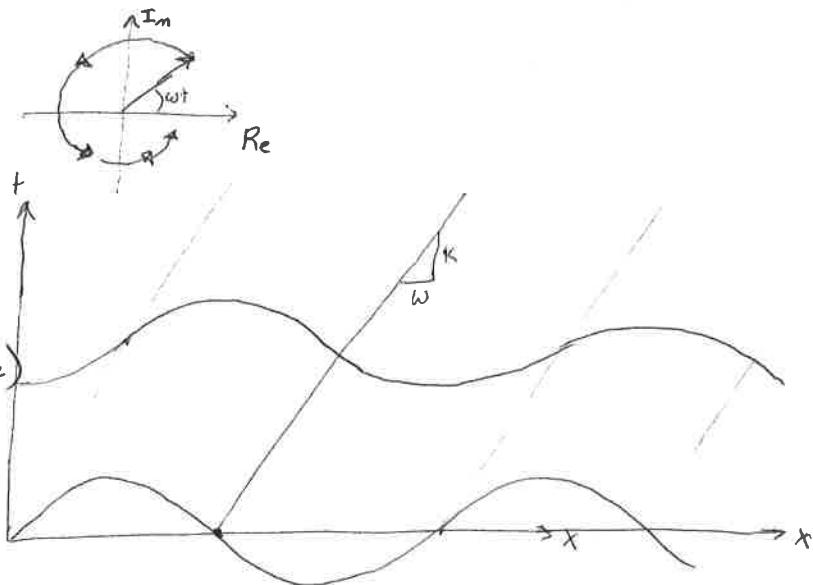
$$U = A e^{i\omega t}$$

$$U = A e^{i(kx - \omega t)}$$

For const value (ie phase)

$$kx - \omega t = C_1$$

$$t = \frac{k}{\omega}x - C_2$$

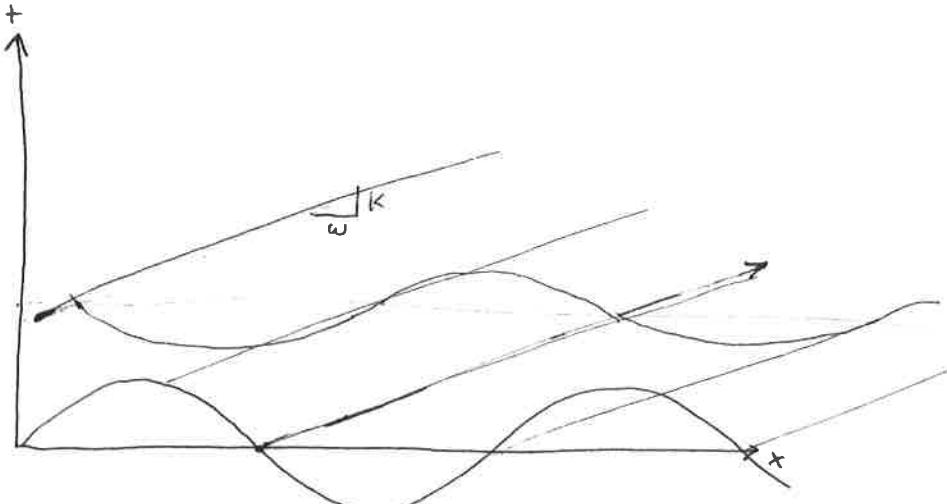


Now for a case
where $\omega > k$

For a constant phase



$$C = \frac{\omega}{k}$$



Group Velocity and Phase Velocity.

- Phase Velocity

Traveling wave

$$C_p = \frac{\omega}{k}$$

- Group Velocity

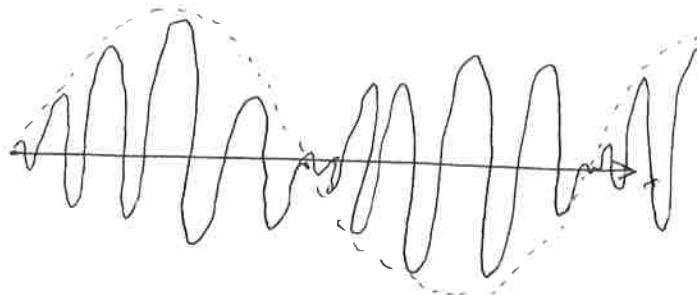
Add two nearby frequencies for beat effect

$$y_1 = A e^{i((k+\Delta k)x - (\omega + \Delta\omega)t)}$$

$$y_2 = A e^{i((k-\Delta k)x - (\omega - \Delta\omega)t)}$$

$$\text{Re}(y_{1+2}) = 2A \underbrace{\cos(\Delta k x - \Delta\omega t)}_{\text{group envelope}} \cdot \underbrace{\cos(kx - \omega t)}_{\text{baseline traveling wave}}$$

$$\sin(A \pm B) = \underline{\sin A \cos B} \pm \underline{\cos A \sin B}$$

Q: Speed of $\cos(akx - \omega t)$? phase speed $C_p = \frac{\omega}{k}$

Q: Speed of Group $\cos(\Delta k x - \Delta\omega t)$? Group speed is $C_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$

- Energy and information transfer operates at the group speed

Phase and Group Speed

- Wave equation

$$U_{tt} = C^2 U_{xx}$$

Apply Fourier/harmonic approximation $U = \Psi(x) e^{i\omega t}$

$$i^2 \omega^2 e^{i\omega t} \Psi(x) = C^2 \Psi_{xx} e^{i\omega t} \Rightarrow$$

Solve for Ψ

$$\Psi = A e^{i(\frac{\omega}{c}x + \omega t)} + B e^{-i(\frac{\omega}{c}x + \omega t)}$$

Solve for U

$$U = \underbrace{A e^{i(\frac{\omega}{c}x + \omega t)}}_{\text{left running}} + \underbrace{B e^{i(-\frac{\omega}{c}x + \omega t)}}_{\text{right running}} = A e^{i\omega(\frac{x}{c} + t)} + B e^{i\omega(-\frac{x}{c} + t)}$$

And...

$$k = \text{term in front of } x = \frac{\omega}{c}$$

Phase speed

$$C_p = \frac{\omega}{k} = \omega \cdot \frac{1}{\frac{\omega}{c}} = \omega \cdot \frac{c}{\omega} = c$$

Group speed

$$C_g = \frac{d\omega}{dk} = \frac{d}{dk}(ck) = c$$

for the wave eqn
 $C_p = C_g$

Beams

$$U_{tt} = \frac{EI}{\rho} U_{xxxx}$$

$$\omega^2 = \frac{EI}{\rho} k^4 \Rightarrow \omega = \sqrt{\frac{EI}{\rho}} k^2 \leftarrow \text{high wavenumbers move fast!}$$

Phase velocity

$$C_p = \frac{\omega}{k} = \sqrt{\frac{EI}{\rho}} \frac{k^2}{k} = \sqrt{\frac{EI}{\rho}} k$$

Group Velocity

$$C_g = \frac{d\omega}{dk} = \frac{d}{dk}(\omega) = \frac{d}{dk}(\sqrt{\frac{EI}{\rho}} k^2) = 2\sqrt{\frac{EI}{\rho}} k = 2C_p$$

In a beam, the group velocity is twice the phase velocity

In General (see Moretti)

$$C_p \propto \lambda^n \quad \text{then} \quad C_g = (1-n) C_p$$

Example:

Surface wave in deep water

$$C_p \propto \lambda^{1/2} \Rightarrow C_g = (1 - \frac{1}{2}) C_p = \frac{1}{2} C_p$$

We notice the effect in videos of water waves.

Tsunami (Japanese: Tsu (harbor) + nami (wave))

Model with the shallow water equations (ironic)

Conservation of mass and momentum in 1D

$$\frac{du}{dt} + u \frac{du}{dx} + g \frac{dh}{dx} = 0$$

Find soliton (solitary wave)

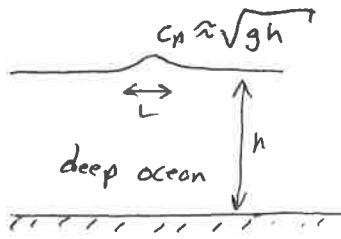


For the related KdV equation

$$C_p = C_g = \sqrt{g(h+H)}$$

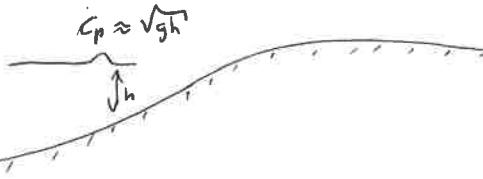
↑ wave height
water depth

Conservation of Mass _{Momentum} is the killer.



$h \approx$

- Mass $\approx H \cdot L$
- Momentum $\approx \frac{1}{2} \rho V^2 = \frac{1}{2} \rho g h$



- h goes to zero
- Speed goes to $C_p = \sqrt{gH}$ slower
- Match momentum $\frac{1}{2} \rho g h = \frac{1}{2} \rho g H$
obviously not correct

Tsunami is not a pure soliton.