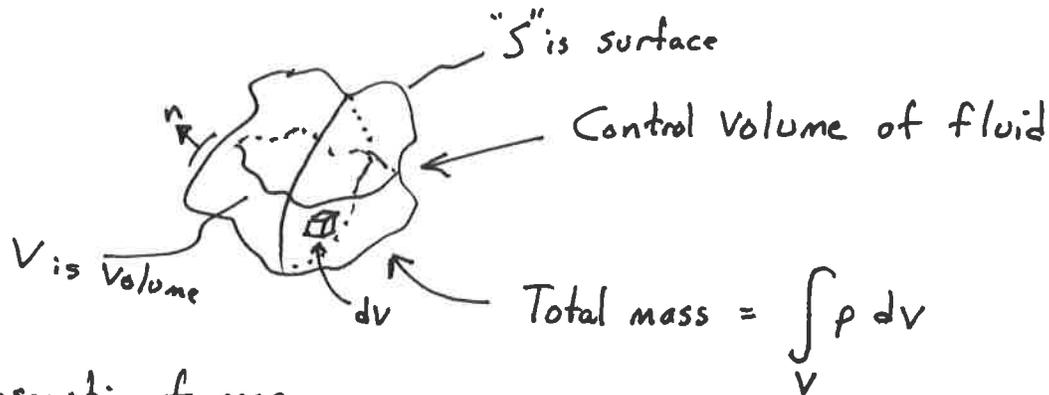


# Fluid Dynamics + Aerodynamics.



- Conservation of mass

$$\int_S \rho \mathbf{V} \cdot \mathbf{n} dS = - \int_V \rho_t dV$$

- We will consider Steady Flows,  $\rho_t = 0$
- Also, apply the divergence identity to change a surface integral into a volume integral

$$\int_S \rho \mathbf{V} \cdot \mathbf{n} dS = \int_V \text{div } \rho \mathbf{V} dV$$

So if  $\int_V \text{div } \rho \mathbf{V} = 0$  and the volume is arbitrary,

$$\boxed{\text{div } \rho \mathbf{V} = 0}$$

compressible

- If the fluid is incompressible (low Mach #), density is relatively unchanged.

$$\boxed{\text{div } \mathbf{V} = 0}$$

incompressible

- Expand  $\text{div } \rho V = 0$  in cartesian coordinate frame

$$\text{div } \rho V = \frac{d}{dx}(\rho U) + \frac{d}{dy}(\rho V) = 0 \quad \text{since } \begin{matrix} u = V_x \\ v = V_y \end{matrix}$$

- Let a "streamfunction"  $\psi$  be defined as

$$\rho U = \frac{d\psi}{dy} \quad \text{and} \quad \rho V = -\frac{d\psi}{dx}$$

Substitute to give

$$\text{div } \rho V = \frac{d}{dx}\left(\frac{d\psi}{dy}\right) + \frac{d}{dy}\left(-\frac{d\psi}{dx}\right) = 0 \quad \checkmark$$

$$\frac{d^2\psi}{dx dy} + -\frac{d^2\psi}{dx dy} = 0$$

- Let a "Velocity Potential"  $\phi$  be defined as

$$\rho U = \frac{d\phi}{dx} \quad \text{and} \quad \rho V = \frac{d\phi}{dy}$$

Substitute to give

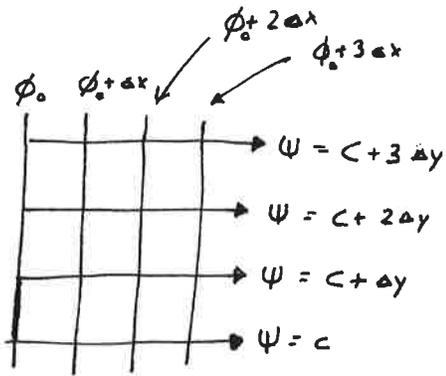
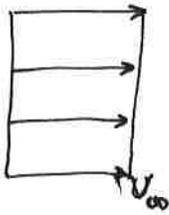
$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$$

This is the Laplacian.

# Stream function $\psi$

# Velocity Potential $\phi$

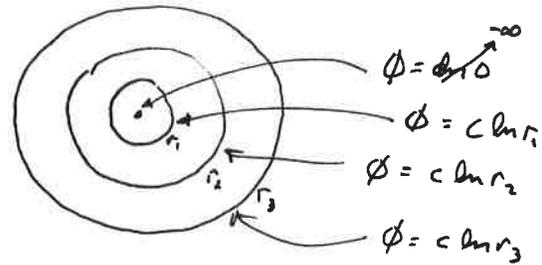
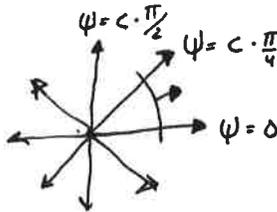
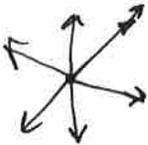
## • Uniform Flow



$$\psi = V_{\infty} y$$

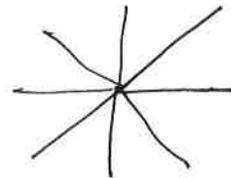
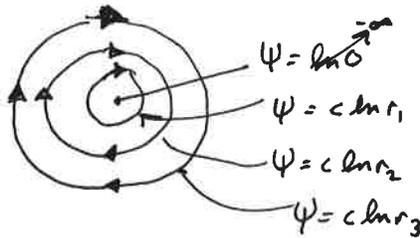
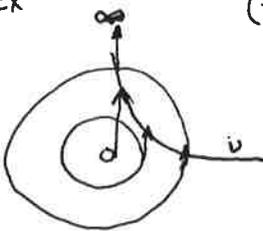
$$\phi = V_{\infty} x$$

## • Point Source

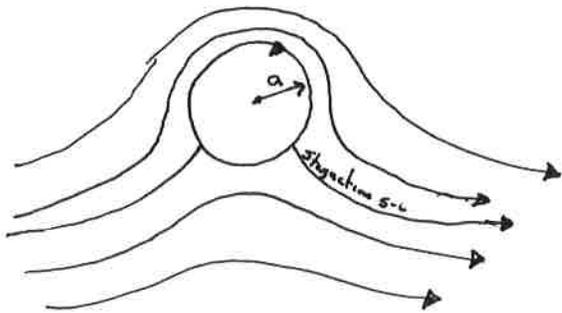


## • Vortex

(think of this as a conjugate image to the pt source)  $\phi$  and  $\psi$  flip



## • Flow past a cylinder with circulation (Lift)



$$\psi = V_{\infty} y \left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{a}\right)$$

$$\phi = \underbrace{V_{\infty} x}_{\text{uniform}} \underbrace{\left(1 + \frac{a^2}{r^2}\right)}_{\text{doublet}} - \underbrace{\frac{\Gamma}{2\pi} \theta}_{\text{vortex}}$$

How much lift?

$$L_{2D} = \rho V \Gamma$$

Show that a theoretical vortex is a solution ~~to~~ to  $\nabla^2 \phi = 0$

$$u = \frac{d\phi}{dx} \quad v = \frac{d\phi}{dy} \quad \phi = -\frac{\Gamma}{2\pi} \theta = -\frac{\Gamma}{2\pi} \arctan\left(\frac{y}{x}\right)$$

$$\nabla^2 \phi = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \quad (\text{using symbolic algebra system}) \quad \text{Messy.}$$

Or the quick way (polar coordinates)  $\phi = -\frac{\Gamma}{2\pi} \theta$

$$\nabla^2 u = \underbrace{u_{rr}}_{\substack{\uparrow \\ \text{no } r \text{ in } \phi}} + \frac{1}{r} u_{r\theta} + \frac{1}{r^2} u_{\theta\theta} = 0$$

$u_{\theta} = -\frac{\Gamma}{2\pi}$   
 $u_{\theta\theta} = 0$

§

Technically, this is NOT true at  $r=0$ . In fact, there is a "residue". From theory, we stay away from the singularity and add a certain term to integrals. (see complex variables)

These potential functions are often called "Harmonic Functions"

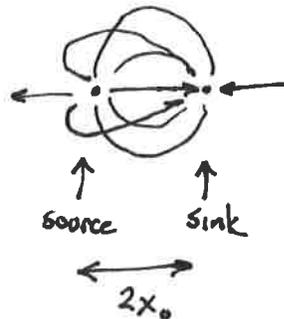
If a mathematician says "check if  $\phi$  is harmonic.", he means  $\nabla^2 \phi = 0$ .

Example.

Verify that the source-sink pair is harmonic.

$$\phi = \frac{\Delta}{4\pi} \ln \frac{(x+x_0)^2 + y^2}{(x-x_0)^2 + y^2}$$

check with Sympy...



• What about Viscous Flows?

Define vorticity as  $\omega = \frac{dv}{dx} - \frac{du}{dy}$

Substitute into the momentum conservation equation (left to you)

$$\frac{d\omega}{dt} + u \frac{d\omega}{dx} + v \frac{d\omega}{dy} = \nu \nabla^2 \omega$$

Combine with  $\psi$  to give

$$\omega = -\nabla^2 \psi$$

Amazing! We can solve viscous incompressible fluid flows with Laplacian analysis.

If inviscid,  $\omega = 0$ . Thus  $\nabla^2 \psi = 0$

But how much  $\Gamma$ ? Good Question

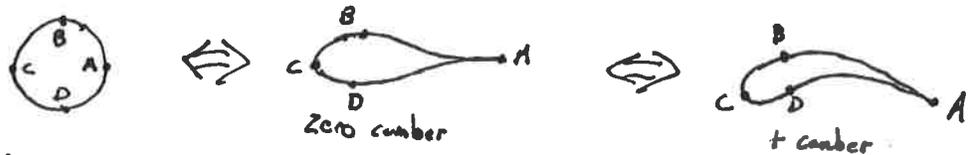
Answering this question takes some theory and experimental concepts.

- Conformal Mapping via Joukowski transforms. ([tiny.cc/GE5554-conformal](http://tiny.cc/GE5554-conformal))

- Convert a circle to an "airfoil" shape with

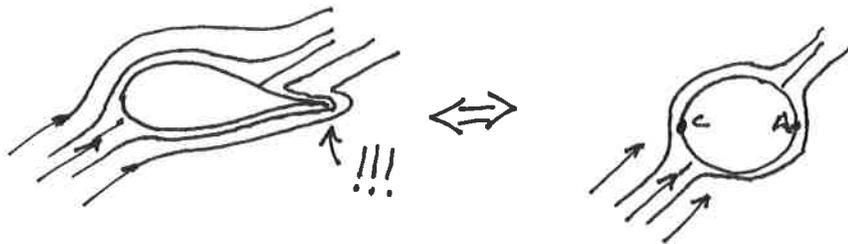
$$Z' = Z + \frac{\lambda^2}{Z} \quad \text{where } Z \text{ is complex } Z = x + iy$$

by moving the center of the circle, we can get an airfoil thickness and camber.



So flow solutions to the circle are mapped to the airfoil

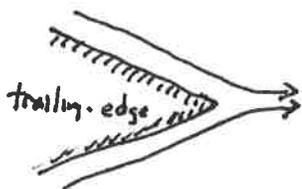
- Joukowski airfoil with angle of attack but No lift



- Kutta Condition

"The flow past a sharp trailing edge leaves the body tangentially to the surface." (paraphrased)

This is backed up by experiments and by ~~inviscid~~ <sup>viscous</sup> flow theory.



We see this



infinite acceleration tends to separate.

NOT This.

- Solution

Pick  $\Gamma$  to ensure the Kutta Condition ( $\Gamma$  is unique)

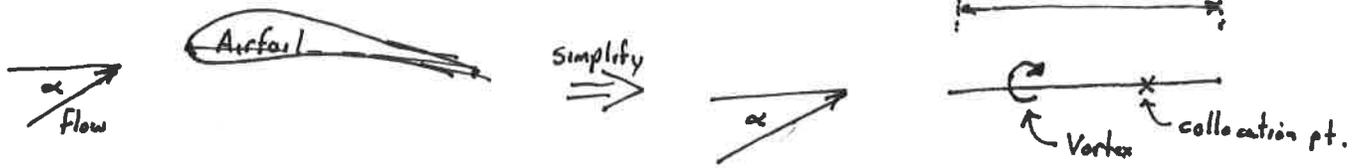
Magnus Effect.

yt demo.

[tiny.cc/GES554-magnus](https://tiny.cc/GES554-magnus)

Baseball "curve balls"

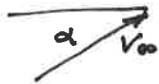
# 2D Panel Method for Airfoils



We can approximate the flow (and lift) of an airfoil section by a freestream and a vortex.

## 1 - segment

- Freestream  $V_\infty$



contribution to vertical component is  $V_\infty \sin \alpha$

- Vortex

⊙

$$\downarrow v = \frac{d\phi}{dy} = \frac{d}{dy} \left( -\frac{\Gamma}{2\pi} \arctan\left(\frac{y}{x}\right) \right) \Rightarrow v(x, 0) = -\frac{\Gamma}{2\pi} \frac{1}{x}$$

$$v_\theta = \nabla \phi = -\frac{\Gamma}{2\pi} \frac{1}{r}$$

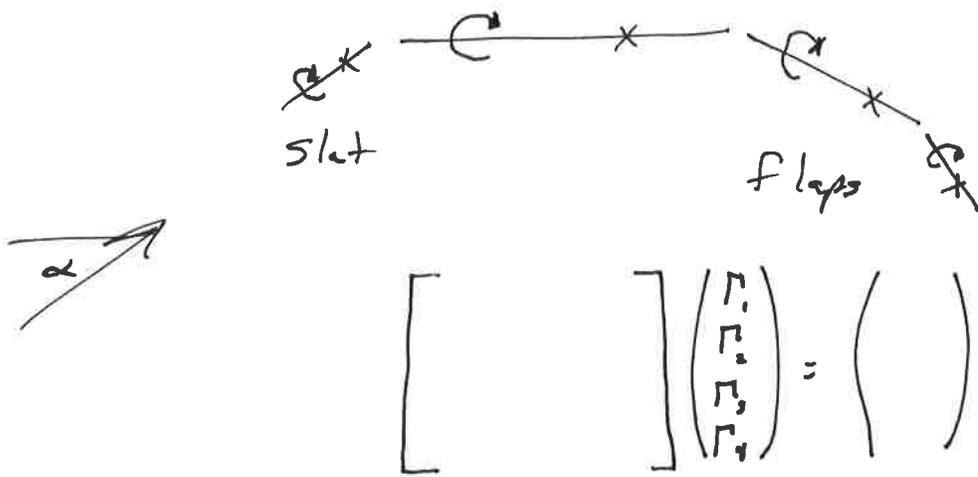
Match the vertical components for zero flow through the airfoil at 75% chord.

$$V_\infty \sin(\alpha) - \frac{\Gamma}{2\pi} \left( \frac{1}{0.75c} \right) = 0 \Rightarrow \Gamma = \pi c V_\infty \sin \alpha$$

Lift<sub>∞</sub> is  $\rho V \Gamma$ . Nondim Lift is  $c_l = \frac{L}{\frac{1}{2} \rho V^2 c}$ ,  $L$  is in  $\alpha$  direction

$$c_l = \frac{\rho V \pi c V_\infty \sin \alpha}{\frac{1}{2} \rho V^2 c} = 2\pi \sin \alpha$$

Lift is  $2\pi \alpha$



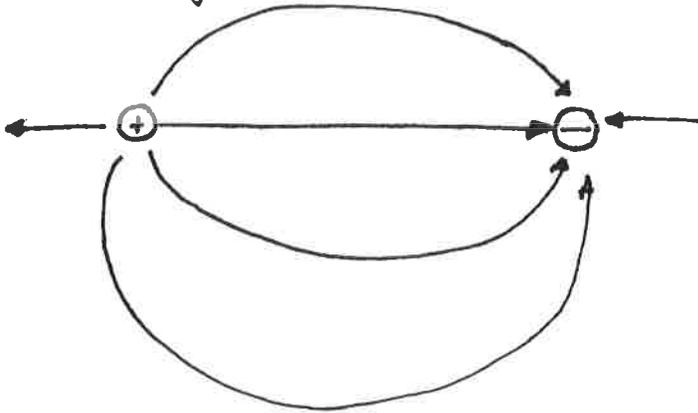
If this is a topic of interest to you, the following classes will discuss this in more detail.

AEM 500

AEM 614

Electrostatics .

pt Charges



We can use our source as a  $\oplus$  and a negative source "sink" as  $\ominus$

potential is

$$\Phi = \frac{\Delta}{4\pi} \ln(x^2 + y^2) = \frac{\Delta}{2\pi} \ln(r)$$

Magnetic Field

