L15P1

$U_t = U_{xx} - 2U_x$

$U(x,0) = \sin(x)$


Visually identify diffusion terms $U_{xx}$ and convection terms $-2U_x$

Strategy: Transform from x coordinate to a new $\xi$ coordinate traveling at velocity 2.

$\xi = x - 2t$

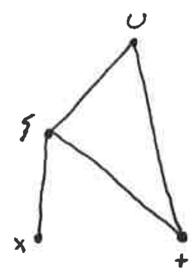

$\frac{du}{dt}$ doesnt change since only x is transformed, right? Charge ahead!

$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{du}{d\xi} \cdot 1 = \frac{du}{d\xi} \Rightarrow \frac{d^2u}{dx^2} = \frac{d^2u}{d\xi^2}$

PDE:

$\frac{du}{dt} = U_{\xi\xi} - 2U_\xi$  Nothing Happened! What is wrong?

Try again, be consistent with transform.




$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$ and $\frac{d^2u}{dx^2} = \frac{d^2u}{d\xi^2}$

$\frac{du}{dt} = \underbrace{\frac{du}{d\xi}\frac{d\xi}{dt}}_{\text{Notice this term!}} + \frac{du}{dt} = \frac{du}{d\xi}(-2) + \frac{du}{dt}$

Subs into PDE

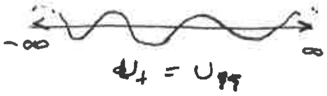
$\frac{du}{dt} - 2\frac{du}{d\xi} = \frac{d^2u}{d\xi^2} - 2\frac{du}{d\xi} \Rightarrow \frac{du}{dt} = \frac{d^2u}{d\xi^2}$

0

So in the $\xi$ direction/coordinate, this is only a diffusion problem.

Use Fourier technique for $\xi = -\infty \leftrightarrow \infty$

$$\overset{\frac{d}{dt}(F(u)=U)}{F(\cancel{u_t})} = \overset{-\omega^2 U}{F(\cancel{u_{\xi\xi}})} \quad \Rightarrow \quad U(t) = e^{-\omega^2 t}\, U_0$$



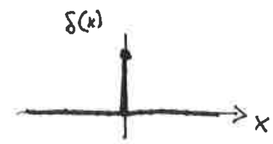

and $F(IC) = F(\sin(x)) = \text{back at book} = i\sqrt{\frac{\pi}{2}}\left(\delta(\omega+1) - \delta(\omega-1)\right)$

$$U = \underbrace{e^{-\omega^2 t}}_{\text{PDE}}\ \underbrace{i\sqrt{\frac{\pi}{2}}\left(\delta(\omega+1) - \delta(\omega-1)\right)}_{\text{IC}}$$

$u = F^{-1}(U)$ but ↑ is certainly not in the book! So messy, right?!

Look at the definition of $F^{-1}(U) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\omega)\, e^{i\omega x}\, d\omega$ This would look like a nasty integral. But we have $\delta$ functions!!

$$u = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\omega^2 t}\, i\sqrt{\frac{\pi}{2}}\Big(\underbrace{\delta(\omega+1)}_{\substack{\text{only has}\\ \text{value when}\\ \omega=-1}} - \underbrace{\delta(\omega-1)}_{\substack{\text{only has}\\ \text{value when}\\ \omega=+1}}\Big)\, e^{i\omega x}\, d\omega$$



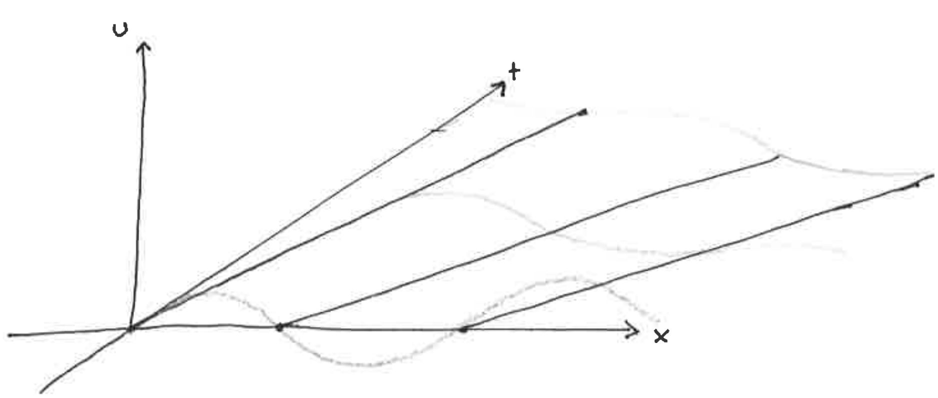

$$= \underbrace{\frac{i}{2} e^{-t} e^{-ix}}_{\omega=-1} \underbrace{- \frac{i}{2} e^{-t} e^{ix}}_{\omega=+1} = \frac{i}{2} e^{-t}\left(e^{-ix} - e^{ix}\right)\cdot\frac{i}{i} = \underbrace{\frac{e^{-t}}{2i}\left(e^{ix} - e^{-ix}\right)}_{\text{Recognize this?!}}$$

$= e^{-t}\sin(x)$ ↖ technically, this is $\xi$. $= e^{-t}\sin(\xi)$

Transform back to $x$ coordinates $x = \xi + 2t$ or $\xi = x - 2t$

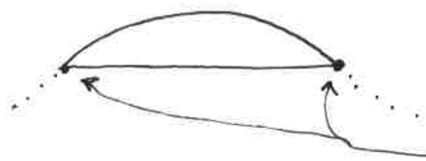
$$\boxed{u = e^{-t}\sin(x-2t)}$$

Translating sine wave that decreases in magnitude in time.



Faster solution

Isolate one cell of sine wave

We know that $\frac{d^2 u}{d\xi^2} = 0$ at the nodal points

Thus, the solution should be the finite domain solution as well.

$$U(\xi, 0) = \sin(\xi) \qquad 0 < \xi < \pi$$

Cell solution

$$U(x,t) = e^{-t}\sin(\xi) \quad \text{in} \quad 0 < \xi < \pi$$

Tile solution $(\pm)$ across domain.

$$U = e^{-t}\sin(\xi)$$

Transform $\xi \to x - 2t$

$$\boxed{U = e^{-t}\sin(x - 2t)}$$

15p2

$U_t = U_{xx} - 2u_x$

$U(x,0) = e^x \sin(x)$

$U_t = U_{xx}$ diffusion and $\underbrace{U_t = -2u_x}$ convection

Solution $\approx$ $U = e^{k_1(x-ct)}$

pull this out of $u(x,t)$

$$U(x,t) = e^{k_1(x-ct)} w(x,t)$$

Subst. into gov equ.

$$\underbrace{-k_1 c\, e^{k_1(x-ct)} w + e^{k_1(x-ct)} w_t}_{U_t} = \underbrace{k_1^2 e^{k_1(x-ct)} w + 2k_1 e^{k_1(x-ct)} w_x + e^{k_1(x-ct)} w_{xx}}_{U_{xx}}$$

$$\underbrace{-2k_1 e^{k_1(x-ct)} w + -2e^{k_1(x-ct)} w_x}_{-2u_x}$$

pull out $e^{k_1(x-ct)}$ term

$-c = \cancel{1-t}^{-1} \Rightarrow c = 1$ $\rightarrow 0$

$$-k_1 c w + w_t = k_1^2 w + 2k_1 w_x + w_{xx} - 2k_1 w - 2w_x .$$

0 if $k_1 = 1$

If $c=1$ and $k_1 = 1$

$$w_t = w_{xx}$$

IC

$$U(x,0) = e^x \sin(x) = e^{\cancel{k_1}^{1}(x-\cancel{ct}^{0})} w(x,0) = e^x w(x,0) \Rightarrow w(x,0) = \sin(x)$$

Solution to $w$

~~wolllwwll~~ $w = e^{-t} \sin(x)$

Combine back to $u = e^{(x-t)} w$

$$U = e^{(x-t)} e^{-t} \sin(x)$$

$$\boxed{u = e^x e^{-2t} \sin(x)}$$

L13P3 $\bar{w}$ Duhamel

$U_t = U_{xx} \qquad 0 < x < \infty$

$U(0,t) = \sin(t)$

$U(x,0) = 0$

$\mathcal{L}(U_t) = \mathcal{L}(U_{xx})$

$sU - \cancel{u(x,0)}^{0} = U_{xx} \qquad \Rightarrow \qquad U = Ae^{-\sqrt{s}x}$

Duhamel's method says to find the impulse response. $u(0,t) = \delta(t)$ so $f(t) = \sin(t)$, $f'(t) = \cos(t)$

$\mathcal{L}(u(0,t)) = \mathcal{L}(\delta(t)) = 1\,\cancel{e^{-s \cdot 0}}^{1} = 1$

$U = e^{-\sqrt{s}\,x}$

$\mathcal{L}^{-1}(U) = \mathcal{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ Not in the book....

$w = \frac{x}{2\sqrt{\pi t}} e^{-x^2/4t}$

Duhamel: $u(x,t) = \int_0^t \underbrace{w_t(x,t-\tau)}_{\text{impulse response}} \underbrace{f(\tau)}_{\text{input}} d\tau$ ← Not as useful as below ($w_t$ is difficult)

and

$= \int_0^t w(x,t-\tau) f'(\tau)\, d\tau + f(0) w(x,t)$ ← we want to use this form

$= \int_0^t \frac{x}{2\sqrt{\pi(t-\tau)}} e^{-\frac{x^2}{4(t-\tau)}} \cos(\tau)\, d\tau + \cancel{f(0)}^{0} w(x,t)$

$$\boxed{u(x,t) = \int_0^t \frac{x}{2\sqrt{\pi(t-\tau)}} e^{-\frac{x^2}{4(t-\tau)}} \cos(\tau)\, d\tau}$$

L6P3

$$u_t = u_{xx} \qquad 0<x<1$$

$$u_x(0,t) = 0$$

$$u_x(1,t) + h\,u(1,t) = 1 \qquad 0<t<\infty$$

$$u(x,0) = \sin(\pi x)$$

The $x=1$ BC is not homogeneous.

- Break into S.S and transient components

$$u = \bar{u} + U \qquad \text{with} \quad \bar{u} = Ax + B$$

1) PDE $\quad \bar{u}_t + U_t = \bar{u}_{xx} + U_{xx} \Rightarrow \boxed{U_t = U_{xx}}$

2) BCs

$$\bar{u}_x(0,t) + U_x(0,t) = 0 \quad \Rightarrow A + U(0,t) = 0 \qquad A = 0$$

$$\boxed{U(0,t) = 0}$$

$$\bar{u}_x(1,t) + U_x(1,t) + h\,\bar{u}(1,t) + h\,U(1,t) = 1$$

$$\cancel{A}^{0} + U_x(1,t) + h(0\cdot 1 + B) + h\,U(1,t) = 1$$

$$U_x(1,t) + hU(1,t) = 1 - hB \quad \Rightarrow \quad B = \frac{1}{h}$$

$$\boxed{U_x(1,t) + hU(1,t) = 0}$$

with $\boxed{\bar{u} = \frac{1}{h}}$

3) IC

$$\bar{u} + U = \sin(\pi x) \quad \Rightarrow \quad \boxed{U = \sin(\pi x) - \frac{1}{h}}$$

L6P3 continued

- Solve new PDE

$$U = XT \quad \text{with} \quad \begin{aligned} &U_t = U_{xx} \\ &U(0,t) = 0 \\ &U_x(1,t) + hU(1,t) = 0 \\ &U = \sin(\pi x) - \frac{1}{h} \end{aligned}$$

- Sep of Vars

$$XT_t = X_{xx}T \Rightarrow \frac{T_t}{T} = \frac{X_{xx}}{X} = -\lambda^2$$

$1^{st}$ order T $\Rightarrow T = T_0 e^{-\lambda^2 t}$

$2^{nd}$ order X $\Rightarrow X = A_m \sin \lambda_m x + B_m \cos \lambda_m x$

- Apply BCs

$$U(0,t) = 0 = X(0)T(t) \Rightarrow X(0) = 0 = A_m \sin 0 + B_m \overset{1}{\cancel{\cos 0}}$$

Thus $B_m = 0$

$$U_x(1,t) + hU(1,t) = 0 = X_x(1,t) + hX(1,t)$$

$$= \lambda A_m \cos \lambda_m \overset{1}{\cancel{x}} + hA_m \sin \lambda_m \overset{1}{\cancel{x}}$$

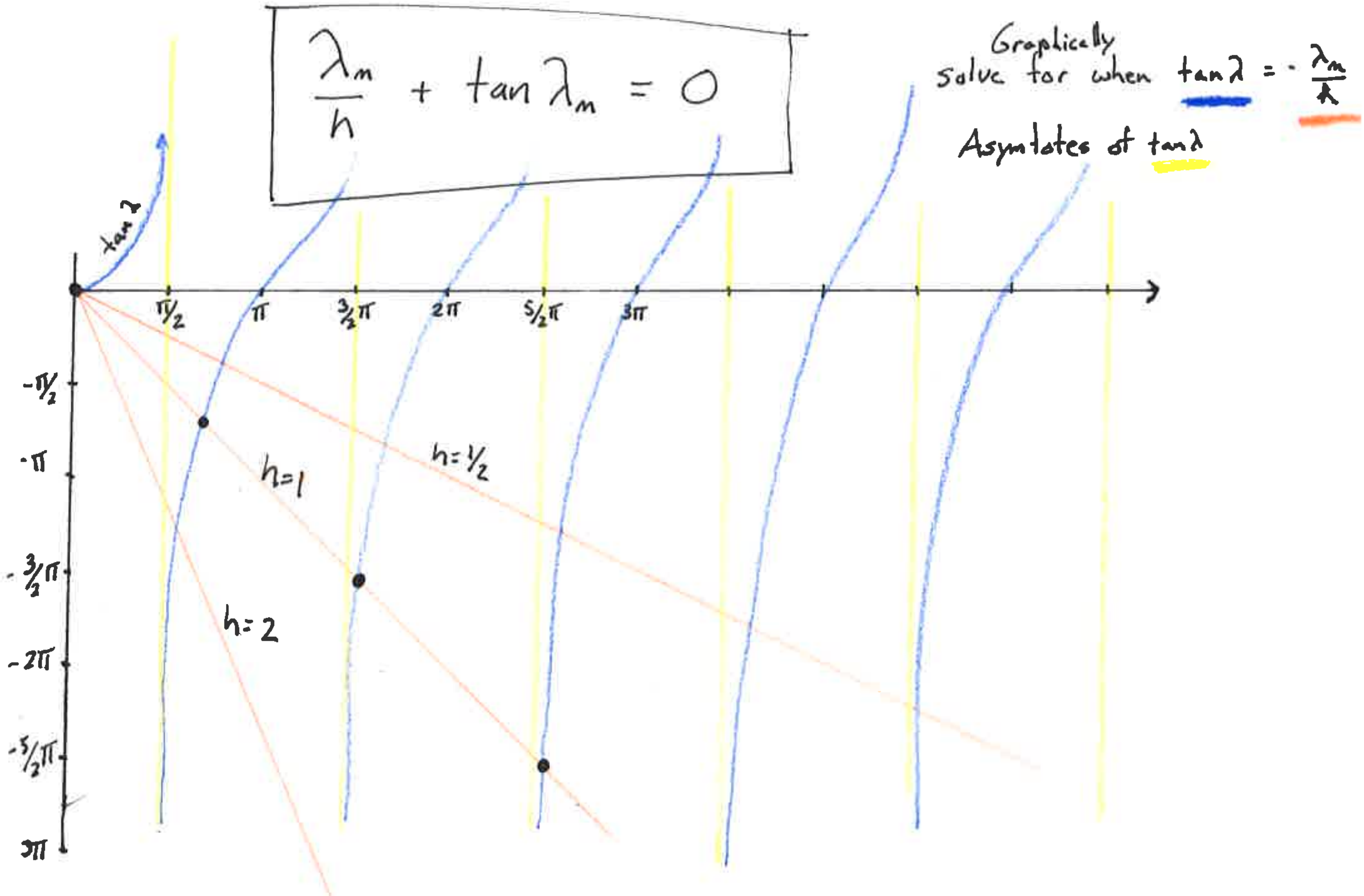
$$= A_m(\lambda_m \cos \lambda_m + h \sin \lambda_m)$$

So, either $\underbrace{A_m = 0}_{\text{boring solution}}$ or $\lambda_m \cos \lambda_m + h \sin \lambda_m = 0$

- Find $\lambda_m$

$$\lambda_m \cos \lambda_m + h \sin \lambda_m = 0$$

$$\boxed{\frac{\lambda_m}{h} + \tan \lambda_m = 0}$$




- Find $\lambda_m$ when $h=1$

| | Exact |
|---|---|
| $\lambda_0 = 0$ useless since $\sin 0 = 0$ | 0 |
| $\lambda_1 \approx$ halfway between $\pi/2$ and $\pi$ $\approx \frac{3}{4}\pi \approx 2.3$ | 2.02876 |
| $\lambda_2 \approx$ asymtote at $3/2\pi \approx 4.7$ | 4.91318 |
| $\lambda_m \approx m_{th}$ asymtote of $\tan\lambda \approx \left(\frac{2m-1}{2}\right)\pi$ | |

- Solution

$$X = A_m \sin(\lambda_m x)$$

$$= A_0 \sin 0 + A_1 \sin(2.3x) + A_2 \sin(4.7x) + \cdots + A_m \sin\left(\tfrac{2m-1}{2}\pi x\right)$$

- Find $A_m$ terms.

$$\phi = A_m \sin(\lambda_m x)$$

premultiply by $\sin(\lambda_n x)$

Integrate over domain $\int_0^1 \cdots dx$

$$\int_0^1 \phi(x) \sin(\lambda_n x)\, dx = A_m \int_0^1 \sin(\lambda_m x) \sin(\lambda_n x)\, dx$$

$$A_m = \frac{\int_0^1 \left(\sin \pi x - \frac{1}{h}\right) \sin(\lambda_n x)\, dx}{\int_0^1 \sin(\lambda_m x) \sin(\lambda_n x)\, dx}$$

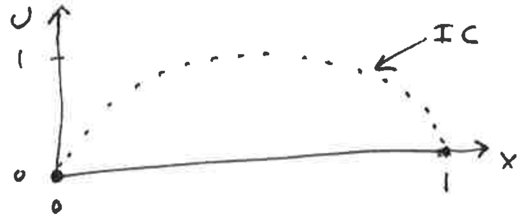
From S-L theory, this only is non-zero when $n=m$!

- Solution

$$\boxed{U = \bar{U} + \mathcal{U} = \frac{1}{h} + A_1 e^{-4.113t} \sin(2.02x) + A_2 e^{-24.14t} \sin(4.7x) + \cdots \quad \text{where } A_m = \frac{\int_0^1 \left(\sin(\pi x) - \frac{1}{h}\right) \sin(\lambda_m x)\, dx}{\int_0^1 \sin^2(\lambda_m x)\, dx}}$$

L9 p1

$$U_t = U_{xx} + \sin(3\pi x)$$
$$U(0,t) = 0$$
$$U(1,t) = 0$$
$$U(x,0) = \sin(\pi x)$$




This is a non-homogeneous PDE ($\sin 3\pi x$). Call $f_n = \sin 3\pi x$

- Find eigenfunctions by solving homogeneous PDE

$$\begin{array}{l} U_t = U_{xx} \\ U(0) = 0 \\ U(1) = 0 \end{array} \quad \xRightarrow{\text{by inspection}} \quad U = XT \text{ with } X = \sin(n\pi x)$$

these are our building blocks....

- Expand non-homogeneous PDE with

$$U = T_n(t)\, X(x)$$

$$T_{n_t} X + T_n X_t = T_{n_{xx}} X + T_n X_{xx} + f_n$$

- Expand $f_n$ in terms of $X$: $\quad f_n = \sin(n\pi x)$

$$\int_0^1 \underset{\substack{\uparrow \\ \sin(3\pi x)}}{f_n} \sin(n\pi x)\, dx = \int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx$$

Obvious?

$$f_n = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_n = 0 \cdot \sin(\pi x) + 0 \cdot \sin(2\pi x) + 1 \cdot \sin(3\pi x) + \cdots$$

- Continue with PDE

$$T_{n_t} X + \cancelto{0}{T_n X_t} = \cancelto{0}{T_{n_{xx}}} X + T_n \overbrace{\cancel{X_{xx}}}^{-n^2\pi^2 \sin(n\pi x) = -n^2\pi^2 X} + \begin{Bmatrix} 1 & n=3 \\ 0 & \text{otherwise} \end{Bmatrix} X$$

- BCs

$$U(0,t) = 0 = T_n(t)\, \cancelto{0}{X(0)}$$
$$U(1,t) = 0 = T_n(t)\, \cancelto{0}{X(1)}$$

$\Rightarrow$ Not useful. Why? $X$!

- ICs

$$U(x,0) = \sin(\pi x) = T_n(0)\, X(x)$$

In general, premultiply by $X$ and integrate to find $T_n(0)$

$$\int_0^1 X \sin(\pi x)\,dx = \int_0^1 T_n(0)\, X^2(x)\,dx \qquad \text{with } X(x) = \sin(n\pi x)$$

$$T_n(0) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

- Solve for $T_n(t)$ in PDE

$$X\left[ T_{n_t}(t) = -n^2\pi^2 T_n(t) + \begin{Bmatrix} 1 & n=3 \\ 0 & \text{otherwise} \end{Bmatrix} \right]$$

Since $X$ is not always zero, the bracket terms must be.

$$T_{n_t} + n^2\pi^2 T_n = \begin{cases} 1 & n=3 \\ 0 & \text{otherwise} \end{cases}$$

$$T_n(0) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

$n=1$

$$\begin{array}{l} T_{n_t} + n^2\pi^2 T_n = 0 \\ T_n(0) = 1 \end{array} \Rightarrow T_n = T_n(0)\, e^{-n^2\pi^2 t} = e^{-n^2\pi^2 t}$$

$n=2$

$$\begin{array}{l} T_{n_t} + n^2\pi^2 T_n = 0 \\ T_n(0) = 0 \end{array} \Rightarrow T_n = \cancel{T_n(0)}^{\,0}\, e^{-n^2\pi^2 t} = 0$$

$n=3$

$$T_{n_t} + n^2\pi^2 T_n = 1$$

$$T_n(0) = 0$$

premultiply by $e^{n^2\pi^2 t}$ $\Rightarrow$ $\underbrace{e^{n^2\pi^2 t} T_{n_t} + e^{n^2\pi^2 t} T_n}_{\substack{\text{product rule} \\ \text{of} \\ \frac{d}{dt}\left(e^{n^2\pi^2 t} T_n\right)}} = e^{n^2\pi^2 t}$

integrate wrt time

$$\underbrace{\int_0^t \frac{d}{dt}\left(e^{n^2\pi^2 t} T_n\right)} = \int_0^t e^{n^2\pi^2 t}\, dt$$

$$e^{n^2\pi^2 t} T_n = \left.\frac{e^{n^2\pi^2 t}}{n^2\pi^2}\right|_0^t = \frac{e^{n^2\pi^2 t} - 1}{n^2\pi^2}$$

$$T_n = \frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2}$$

$n=4$ $\quad T_n = 0$

- Solution

$$U = e^{-1^2\pi^2 t} \sin(1\pi x) + 0 \cdot \sin(2\pi x) + \frac{1 - e^{-3^2\pi^2 t}}{3^2\pi^2} \sin(3\pi x) + 0 \cdots$$

$$\boxed{U = e^{-\pi^2 t} \sin(\pi x) + \frac{1 - e^{-9\pi^2 t}}{9\pi^2} \sin(3\pi x)}$$

L13P2 "Tricky"

$$U_t = \alpha^2 U_{xx} \qquad -\infty < x < \infty$$

$$U(x,0) = \sin x$$

$$L(U_t) = L(\alpha^2 U_{xx}) \Rightarrow sU - \cancel{u(x,0)}^{\sin x} = \alpha^2 U_{xx} \Rightarrow U_{xx} - \frac{s}{\alpha^2}U = -\frac{\sin x}{\alpha^2}$$

We know that $U$ is composed of a $\underbrace{\text{homogeneous solution}}_{e^{-\frac{\sqrt{s}}{\alpha}x} \quad e^{\frac{\sqrt{s}}{\alpha}x}}$ and a $\underbrace{\text{particular solution}}_{P\,?}$

$$U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + \underbrace{B\cancel{e^{\frac{\sqrt{s}}{\alpha}x}}^{0}}_{\text{must be bounded}} + C\cdot P(x)$$

find a $p(x)$....

expect $p(x)$ to look like $\sin(x)\cdot(\text{terms}) \Rightarrow p(x) = \sin(x)\,Z(x)$

Substitute $U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + C\,p(x)$ into ODE

$$\underbrace{A\frac{s}{\alpha^2}e^{-\frac{\sqrt{s}}{\alpha}x} + C(-Z)\sin(x) - \frac{s}{\alpha^2}\Big(Ae^{-\frac{\sqrt{s}}{\alpha}x}}_{0 \text{ allways cancel...}} + CZ\sin(x)\Big) = -\frac{\sin(x)}{\alpha^2}$$

Remaining terms.

$$-ZC\sin(x)\left(1 + \frac{s}{\alpha^2}\right) = -\frac{\sin(x)}{\alpha^2}$$

Since $\sin(x)$ is not always zero...

$$ZC = \frac{1}{\alpha^2}\,\frac{1}{1+\frac{s}{\alpha^2}} = \frac{1}{\alpha^2+s}$$

Solution to $U$

$$U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + \frac{1}{\alpha^2+s}\sin(x)$$

Inverse Laplace transform is linear

$$\mathcal{L}^{-1}(A+B) = \mathcal{L}^{-1}(A) + \mathcal{L}^{-1}(B) \quad \text{and} \quad \mathcal{L}^{-1}\left(Ae^{-\frac{\sqrt{s}}{\alpha}x}\right) = \mathcal{L}^{-1}\left(s\cdot\frac{e^{-\frac{\sqrt{s}}{\alpha}x}}{s}\right) = \text{complicated}$$

$$\mathcal{L}^{-1}\left(\frac{\sin(x)}{\alpha^2+s}\right) = \sin(x)\,\mathcal{L}^{-1}\left(\frac{1}{\alpha^2+s}\right) = \sin x\, e^{-\alpha^2 t}$$

Fit the IC. $u(x,0) = \sin(x)$

$$U = A\cdot\text{complicated} + \sin(x)\,e^{-\alpha^2 t}$$

Only this term fits ICs $\Rightarrow$ $\boxed{U = \sin(x)\,e^{-\alpha^2 t}}$

L13p3

$$U_t = U_{xx} \qquad 0 < x < \infty$$
$$U(0,t) = \sin(t) \qquad 0 < t < \infty$$
$$U(x,0) = 0 \qquad 0 \le x < \infty$$


Apply Laplace transform to ~~$x$~~ $t$ (BC has time term)

$$\mathcal{L}(U_t) = \mathcal{L}(U_{xx})$$

$$sU(x) - u(x,0) = \frac{d^2U(x)}{dx^2}$$

Apply IC and simplify to canonical form

$$sU(x) = \frac{d^2U}{dx^2} \Rightarrow U_{xx} - sU = 0$$

Apply $\mathcal{L}$ to BC

$$\mathcal{L}(u(0,t)) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1} = U(0)$$

The solution to the canonical form is

$$U = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x}$$

Notice that since $u$ at $x=\infty$ must be bounded, the $e^{\sqrt{s}x}$ term must make the $B$ term equal $0$.

$$U = Ae^{-\sqrt{s}x} = U(0)e^{-\sqrt{s}x}$$

Apply BC at $x=0$

$$U = \frac{1}{s^2+1}e^{-\sqrt{s}x}$$

Apply $\mathcal{L}^{-1}$. Find transforms in appendix.

1) $\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{s}a}}{s}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$ ← to get the $\frac{1}{s}$, mult and divide by $s$ and get $\frac{s}{s^2+1}$

2) $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t)$

Apply Convolution since we have $\mathcal{L}^{-1}$ of two parts

$$\mathcal{L}^{-1}\left(\mathcal{L}(f)\,\mathcal{L}(g)\right) = f * g$$ $\leftarrow$ compose, NOT "times"

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

So our solution is

$$\boxed{u(x,t) = \int_0^t \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{1}{\tau}}\right)\cos(t-\tau)\,d\tau}$$