L6P3

$$u_t = u_{xx} \qquad 0<x<1$$

$$u_x(0,t) = 0$$

$$u_x(1,t) + h\,u(1,t) = 1 \qquad 0<t<\infty$$

$$u(x,0) = \sin(\pi x)$$

The $x=1$ BC is not homogeneous.

- Break into S.S and transient components

$$u = \bar{u} + U \qquad \text{with} \quad \bar{u} = Ax + B$$

1) PDE $\quad \bar{u}_t + U_t = \bar{u}_{xx} + U_{xx} \quad \Rightarrow \quad \boxed{U_t = U_{xx}}$

2) BCs

$$\bar{u}_x(0,t) + U_x(0,t) = 0 \quad \Rightarrow \quad A + U(0,t) = 0 \qquad A = 0$$

$$\boxed{U(0,t) = 0}$$

$$\bar{u}_x(1,t) + U_x(1,t) + h\,\bar{u}(1,t) + h\,U(1,t) = 1$$

$$\cancel{A}^{\,0} + U_x(1,t) + h\,(0\cdot 1 + B) + h\,U(1,t) = 1$$

$$U_x(1,t) + h\,U(1,t) = 1 - hB \quad \Rightarrow \quad B = \frac{1}{h}$$

$$\boxed{U_x(1,t) + hU(1,t) = 0}$$

with $\boxed{\bar{u} = \frac{1}{h}}$

3) IC

$$\bar{u} + U = \sin(\pi x) \quad \Rightarrow \quad \boxed{U = \sin(\pi x) - \frac{1}{h}}$$

L6P3 continued

- Solve new PDE

$$U = XT \quad \text{with} \quad \begin{aligned} &U_t = U_{xx} \\ &U(0,t) = 0 \\ &U_x(1,t) + hU(1,t) = 0 \\ &U = \sin(\pi x) - \tfrac{1}{h} \end{aligned}$$

- Sep of Vars

$$XT_t = X_{xx}T \quad \Rightarrow \quad \frac{T_t}{T} = \frac{X_{xx}}{X} = -\lambda^2$$

$1^{st}$ order $T \Rightarrow T = T_0 e^{-\lambda^2 t}$

$2^{nd}$ order $X \Rightarrow X = A_m \sin \lambda_m x + B_m \cos \lambda_m x$

- Apply BCs

$$U(0,t) = 0 = X(0)T(t) \quad \Rightarrow \quad X(0) = 0 = A_m \sin 0 + B_m \cancel{\cos 0}^{1}$$

Thus $B_m = 0$

$$\begin{aligned} U_x(1,t) + hU(1,t) = 0 &= X_x(1,t) + hX(1,t) \\ &= \lambda A_m \cos \lambda_m \cancel{x}^{1} + hA_m \sin \lambda_m \cancel{x}^{1} \\ &= A_m(\lambda_m \cos \lambda_m + h \sin \lambda_m) \end{aligned}$$

So, either $\underbrace{A_m = 0}_{\text{boring solution}}$ or $\lambda_m \cos \lambda_m + h \sin \lambda_m = 0$

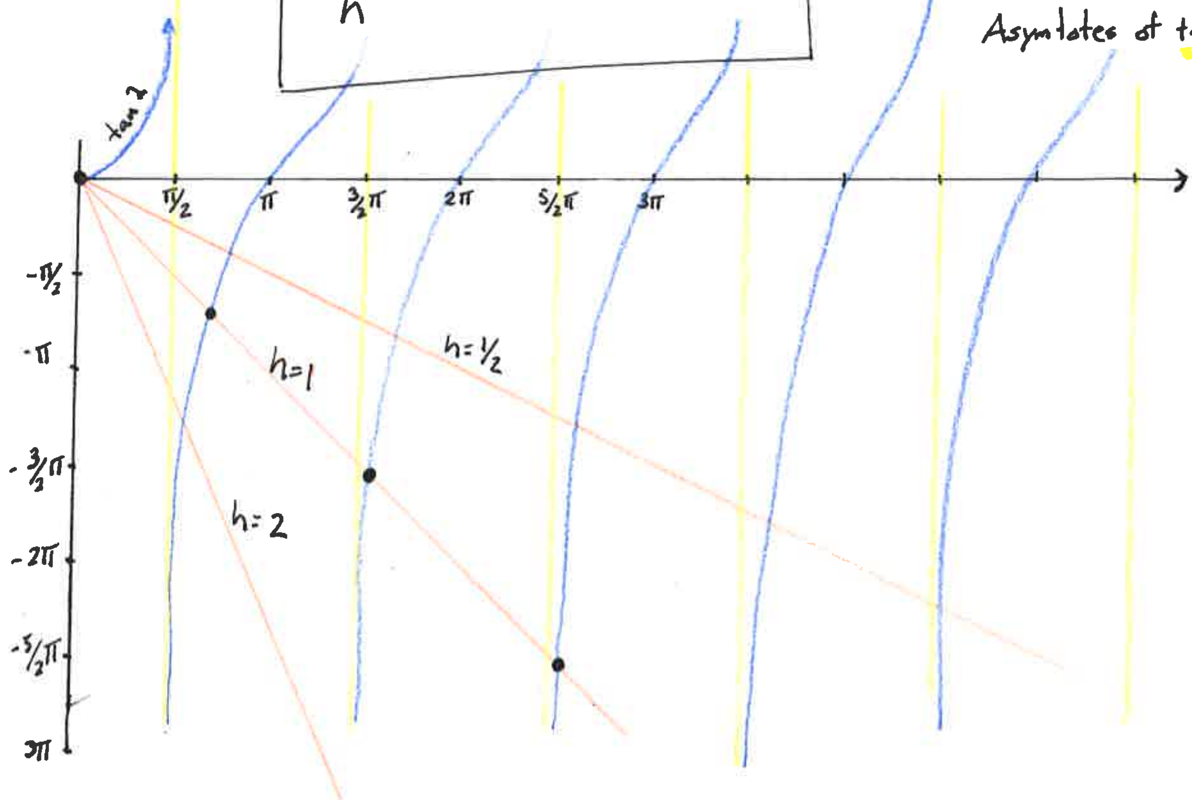
- Find $\lambda_m$

$$\lambda_m \cos \lambda_m + h \sin \lambda_m = 0$$

$$\boxed{\frac{\lambda_m}{h} + \tan \lambda_m = 0}$$

Graphically solve for when $\tan\lambda = -\frac{\lambda_m}{h}$

Asymptotes of $\tan\lambda$




- Find $\lambda_m$ when $h=1$

| | Exact |
|---|---|
| $\lambda_0 = 0$ useless since $\sin 0 = 0$ | 0 |
| $\lambda_1 \approx$ halfway between $\pi/2$ and $\pi \approx \frac{3}{4}\pi \approx 2.3$ | 2.02876 |
| $\lambda_2 \approx$ asymptote at $3/2\pi \approx 4.7$ | 4.91318 |
| $\lambda_m \approx m_{th}$ asymptote of $\tan\lambda \approx \left(\frac{2m-1}{2}\right)\pi$ | |

• Solution

$$X = A_m \sin(\lambda_m x)$$

$$= A_0 \sin 0 + A_1 \sin(2.3x) + A_2 \sin(4.7x) + \cdots + A_m \sin\left(\tfrac{2m-1}{2}\pi x\right)$$

• Find $A_n$ terms.

$$\phi = A_m \sin(\lambda_m x)$$

premultiply by $\sin(\lambda_n x)$

Integrate over domain $\int_0^1 \cdots \, dx$

$$\int_0^1 \phi(x) \sin(\lambda_n x)\, dx = A_m \int_0^1 \sin(\lambda_m x) \sin(\lambda_n x)\, dx$$

$$A_m = \frac{\int_0^1 \left(\sin \pi x - \frac{1}{h}\right) \sin(\lambda_n x)\, dx}{\int_0^1 \sin(\lambda_m x) \sin(\lambda_n x)\, dx}$$

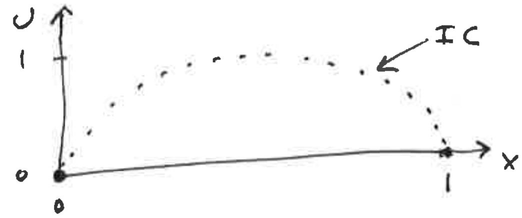
From S-L theory, this only is non-zero when $n=m$!

• Solution

$$\boxed{U = \bar{U} + \tilde{U} = \frac{1}{h} + A_1 e^{-4.113t} \sin(2.02x) + A_2 e^{-24.14t} \sin(4.7x) + \cdots \quad \text{where } A_m = \frac{\int_0^1 \left(\sin(\pi x) - \frac{1}{h}\right) \sin(\lambda_m x)\, dx}{\int_0^1 \sin^2(\lambda_m x)\, dx}}$$

L9 p1

$$U_t = U_{xx} + \sin(3\pi x)$$
$$U(0,t) = 0$$
$$U(1,t) = 0$$
$$U(x,0) = \sin(\pi x)$$




This is a non-homogeneous PDE ($\sin 3\pi x$). Call $f_n = \sin 3\pi x$

- Find eigenfunctions by solving homogeneous PDE

$$\begin{matrix} U_t = U_{xx} \\ U(0) = 0 \\ U(1) = 0 \end{matrix} \quad \overset{\text{by inspection}}{\Longrightarrow} \quad U = XT \text{ with } X = \sin(n\pi x)$$

these are our building blocks....

- Expand ~~solution~~ non-homogeneous PDE with

$$U = T_n(t)\, X(x)$$

$$T_{n_t} X + T_n X_t = T_{n_{xx}} X + T_n X_{xx} + f_n$$

- Expand $f_n$ in terms of $X$: $\quad f_n = \sin(n\pi x)$

$$\int_0^1 \underset{\substack{\uparrow \\ \sin(3\pi x)}}{f_n} \sin(n\pi x)\, dx = \int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx$$

Obvious?

$$f_n = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_n = 0 \cdot \sin(\pi x) + 0 \cdot \sin(2\pi x) + 1 \cdot \sin(3\pi x) + \cdots$$

- Continue with PDE

$$T_{n_t} X + \cancel{T_n X_t}^{\,0} = \cancel{T_{n_{xx}} X}^{\,0} + T_n \underbrace{\cancel{X_{xx}}}_{-n^2\pi^2 \sin(n\pi x) = -n^2\pi^2 X} + \begin{Bmatrix} 1 & n=3 \\ 0 & \text{otherwise} \end{Bmatrix} X$$

- BCs

$$U(0,t) = 0 = T_n(t)\,\cancel{X(0)}^{\,0}$$

$$U(1,t) = 0 = T_n(t)\,\cancel{X(1)}^{\,0} \quad \Rightarrow \quad \text{Not useful. Why? } X!$$

- ICs

$$U(x,0) = \sin(\pi x) = T_n(0)\,X(x)$$

In general, premultiply by $X$ and integrate to find $T_n(0)$

$$\int_0^1 X \sin(\pi x)\,dx = \int_0^1 T_n(0)\,X^2(x)\,dx \qquad \text{with } X(x) = \sin(n\pi x)$$

$$T_n(0) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

- Solve for $T_n(t)$ in PDE

$$X\left[ T_{n_t}(t) = -n^2\pi^2 T_n(t) + \begin{Bmatrix} 1 & n=3 \\ 0 & \text{otherwise} \end{Bmatrix} \right]$$

Since $X$ is not always zero, the bracket terms must be.

$$T_{n_t} + n^2\pi^2 T_n = \begin{cases} 1 & n=3 \\ 0 & \text{otherwise} \end{cases}$$

$$T_n(0) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

$n=1$

$$T_{n_t} + n^2\pi^2 T_n = 0 \qquad \Rightarrow \quad T_n = T_n(0)e^{-n^2\pi^2 t} = e^{-n^2\pi^2 t}$$
$$T_n(0) = 1$$

$n=2$

$$T_{n_t} + n^2\pi^2 T_n = 0 \qquad \Rightarrow \quad T_n = \overset{0}{\cancel{T_n(0)}} e^{-n^2\pi^2 t} = 0$$
$$T_n(0) = 0$$

$n=3$

$$T_{n_t} + n^2\pi^2 T_n = 1$$
$$T_n(0) = 0$$

premultiply by $e^{n^2\pi^2 t}$ $\Rightarrow$ $\underbrace{e^{n^2\pi^2 t}T_{n_t} + e^{n^2\pi^2 t}T_n}_{\text{product rule of } \frac{d}{dt}(e^{n^2\pi^2 t}T_n)} = e^{n^2\pi^2 t}$

integrate wrt time

$$\underbrace{\int_0^t \frac{d}{dt}\left(e^{n^2\pi^2 t}T_n\right)} = \int_0^t e^{n^2\pi^2 t}\,dt$$

$$e^{n^2\pi^2 t}T_n = \left.\frac{e^{n^2\pi^2 t}}{n^2\pi^2}\right|_0^t = \frac{e^{n^2\pi^2 t} - 1}{n^2\pi^2}$$

$$T_n = \frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2}$$

$n=4$ $\quad T_n = 0$

- Solution

$$U = e^{-1^2\pi^2 t}\sin(1\pi x) + 0\cdot\sin(2\pi x) + \frac{1-e^{-3^2\pi^2 t}}{3^2\pi^2}\sin(3\pi x) + 0\ldots$$

$$\boxed{U = e^{-\pi^2 t}\sin(\pi x) + \frac{1-e^{-9\pi^2 t}}{9\pi^2}\sin(3\pi x)}$$

L13P2 "Tricky"

$$U_t = \alpha^2 U_{xx} \qquad -\infty < x < \infty$$

$$U(x,0) = \sin x$$

$$L(U_t) = L(\alpha^2 U_{xx}) \Rightarrow sU - \cancel{u(x,0)}^{\sin x} = \alpha^2 U_{xx} \Rightarrow U_{xx} - \frac{s}{\alpha^2} U = -\frac{\sin x}{\alpha^2}$$

We know that $U$ is composed of a $\underbrace{\text{homogeneous solution}}_{e^{-\frac{\sqrt{s}}{\alpha}x} \quad e^{\frac{\sqrt{s}}{\alpha}x}}$ and a $\underbrace{\text{particular solution}}_{P?}$

$$U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + \cancel{Be^{\frac{\sqrt{s}}{\alpha}x}}^{0} + C \cdot P(x)$$

must be bounded

find a $p(x)$....

expect $p(x)$ to look like $\sin(x) \cdot (\text{terms}) \Rightarrow p(x) = \sin(x)\, Z(x)$

Substitute $U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + C\,p(x)$ into ODE

$$\underbrace{A\frac{s}{\alpha^2}e^{-\frac{\sqrt{s}}{\alpha}x} + C(-Z)\sin(x) - \frac{s}{\alpha^2}\left(Ae^{-\frac{\sqrt{s}}{\alpha}x}\right.}_{0 \text{ always cancel...}} \left. + CZ\sin(x)\right) = -\frac{\sin(x)}{\alpha^2}$$

Remaining terms.

$$-ZC\sin(x)\left(1 + \frac{s}{\alpha^2}\right) = -\frac{\sin(x)}{\alpha^2}$$

Since $\sin(x)$ is not always zero...

$$ZC = \frac{1}{\alpha^2}\frac{1}{1+\frac{s}{\alpha^2}} = \frac{1}{\alpha^2 + s}$$

Solution to $U$

$$U = Ae^{-\frac{\sqrt{s}}{\alpha}x} + \frac{1}{\alpha^2+s}\sin(x)$$

Inverse Laplace transform is linear

$$\mathcal{L}^{-1}(A+B) = \mathcal{L}^{-1}(A) + \mathcal{L}^{-1}(B) \quad \text{and} \quad \mathcal{L}^{-1}\left(Ae^{-\frac{\sqrt{s}}{\alpha}x}\right) = \mathcal{L}^{-1}\left(s \cdot \frac{e^{-\frac{\sqrt{s}}{\alpha}x}}{s}\right) = \text{complicated}$$

$$\mathcal{L}^{-1}\left(\frac{\sin(x)}{\alpha^2+s}\right) = \sin(x)\,\mathcal{L}^{-1}\left(\frac{1}{\alpha^2+s}\right) = \sin x\, e^{-\alpha^2 t}$$

Fit the IC. $u(x,0) = \sin(x)$

$$U = A \cdot \text{complicated} + \sin(x)\, e^{-\alpha^2 t}$$

only this term fits ICs $\Rightarrow$ $\boxed{U = \sin(x)\, e^{-\alpha^2 t}}$

L13p3


$U_t = U_{xx}$ $\quad 0 < x < \infty$

$U(0,t) = \sin(t)$ $\quad 0 < t < \infty$

$U(x,0) = 0$ $\quad 0 \le x < \infty$

Apply Laplace transform to t (BC has time term)

$$\mathcal{L}(U_t) = \mathcal{L}(U_{xx})$$

$$sU(x) - u(x,0) = \frac{d^2U(x)}{dx^2}$$

Apply IC and simplify to canonical form

$$sU(x) = \frac{d^2U}{dx^2} \Rightarrow U_{xx} - sU = 0$$

Apply $\mathcal{L}$ to BC

$$\mathcal{L}(u(0,t)) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1} = U(0)$$

The solution to the canonical form is

$$U = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x}$$

Notice that since $u$ at $x=\infty$ must be bounded, the $e^{\sqrt{s}x}$ term must make the B term equal 0.

$$U = Ae^{-\sqrt{s}x} = U(0)e^{-\sqrt{s}x}$$

Apply BC at $x=0$

$$U = \frac{1}{s^2+1}e^{-\sqrt{s}x}$$

Apply $\mathcal{L}^{-1}$. Find transforms in appendix.

1) $\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{s}a}}{s}\right) = \operatorname{erfc}\left(\frac{a}{2}\frac{1}{\sqrt{t}}\right)$ $\leftarrow$ to get the $\frac{1}{s}$, mult and divide by s and get $\frac{s}{s^2+1}$

2) $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t)$

Apply Convolution since we have $\mathcal{L}^{-1}$ of two parts

$$\mathcal{L}^{-1}\left(\mathcal{L}(f)\,\mathcal{L}(g)\right) = f * g \quad \leftarrow \text{compose, } \underline{\text{NOT}} \text{ "times"}$$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

So our solution is

$$\boxed{u(x,t) = \int_0^t \operatorname{erfc}\left(\frac{x}{2}\frac{1}{\sqrt{\tau}}\right) \cos(t-\tau)\,d\tau}$$