

# Derivation of D'Alembert (Hints)

General Solution

$$U(x,t) = \phi(x-ct) + \psi(x+ct) = \phi(\eta) + \psi(\xi)$$

Initial Conditions

$$U(x,0) = f(x) = \phi(x) + \psi(x) \quad \text{since } t=0$$

$$U_t(x,0) = g(x) = \frac{d\phi}{dt}(\eta) + \frac{d\psi}{dt}(\xi)$$

$$= \frac{d\phi}{d\eta} \frac{d\eta}{dt} + \frac{d\psi}{d\xi} \frac{d\xi}{dt} = \frac{d\phi}{d\eta} \left( \frac{d(x-ct)}{dt} \right) + \frac{d\psi}{d\xi} \left( \frac{d(x+ct)}{dt} \right)$$

$$= \frac{d\phi}{d\eta}(-c) + \frac{d\psi}{d\xi}(+c) \quad \text{but at } t=0, \eta = \xi$$

$$= -c \frac{d\phi}{d\eta} + c \frac{d\psi}{d\xi}$$

$$= -c \frac{d\phi}{dx} + c \frac{d\psi}{dx}$$

Now, 2 equations and 2 unknowns.

Integrate  $\int_{\eta_0}^{\eta} g(\eta) d\eta = \int_{\eta_0}^{\eta} -c \frac{d\phi}{d\eta} + c \frac{d\psi}{d\eta} d\eta$

$$\int_{\eta_0}^{\eta} g(\eta) d\eta = -c \phi(\eta) + c \psi(\eta) + \underbrace{c\phi(\eta_0) - c\psi(\eta_0)}_{\text{constants}}$$

Solve for  $\psi$

$$\psi = \frac{1}{2c} \int_{x_0}^{x'} g(\eta) d\eta + \frac{1}{2} \phi(x',t) - K \quad x' \text{ means } x+ct$$

Solve for  $\phi$

$$\phi = -\frac{1}{2c} \int_{x_0}^{x'} g(\eta) d\eta + \frac{1}{2} \psi(x',t) + K \quad x' \text{ means } x-ct$$

General solution

$$U(x,t) = \psi + \phi = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x_0}^{x+ct} g(\eta) d\eta - \frac{1}{2c} \int_{x'_0}^{x-ct} g(\eta) d\eta - K + K$$

$$U(x,t) = \frac{1}{2} \left( f(x+ct) + f(x-ct) \right) + \frac{1}{2c} \int_{x_0'}^{x+ct} g(\eta) d\eta - \frac{1}{2c} \int_{x_0'}^{x-ct} g(\eta) d\eta$$

+  $\frac{1}{2c} \int_{x-ct}^{x_0'} g(\eta) d\eta$

Combine the integral

$$\int_{x-ct}^{x_0'} + \int_{x_0'}^{x+ct} = \int_{x-ct}^{x+ct}$$

$$U(x,t) = \frac{1}{2} \left( f(x+ct) + f(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\eta) d\eta$$