P	n_1	n_4	n_6	n_{12}	n_{24}	l_1	l_2	l_3	b_4
8	1					0.2500000000	0.2500000000	0.2500000000	0.2500000000
		2				1.0000000000	0.0000000000	0.0000000000	0.0000000000
						0.0991203900	0.0991203900	0.0991203900	0.7026388300
			2			0.5000000000	0.5000000000	0.0000000000	0.0000000000
						0.3920531037	0.3920531037	0.1079468963	0.1079468963
				8		0.9498789977	0.0501210023	0.0000000000	0.0000000000
						0.8385931398	0.1614068602	0.0000000000	0.0000000000
						0.6815587319	0.3184412681	0.0000000000	0.0000000000
						0.0660520784	0.0660520784	0.8678958432	0.0000000000
						0.2033467796	0.2033467796	0.5933064408	0.0000000000
						0.3905496216	0.3905496216	0.2189007568	0.0000000000
						0.1047451941	0.1047451941	0.5581946462	0.2323149656
						0.2419418605	0.2419418605	0.4062097450	0.1099065340
					2	0.3617970895	0.5541643672	0.0840385433	0.0000000000
						0.1801396087	0.7519065566	0.0679538347	0.0000000000
9		4				1.0000000000	0.0000000000	0.0000000000	0.0000000000
						0.3333333333	0.3333333333	0.3333333333	0.0000000000
						0.0823287303	0.0823287303	0.0823287303	0.7530138091
						0.2123055477	0.2123055477	0.2123055477	0.3630833569
				11		0.9597669541	0.0402330459	0.0000000000	0.0000000000
						0.8693869326	0.1306130674	0.0000000000	0.0000000000
						0.7389624749	0.2610375251	0.0000000000	0.0000000000
						0.5826394788	0.4173605212	0.0000000000	0.0000000000
						0.0355775717	0.0355775717	0.9288448566	0.0000000000
						0.4640303025	0.4640303025	0.0719393950	0.0000000000
						0.1633923069	0.1633923069	0.6732153862	0.0000000000
						0.0873980781	0.0873980781	0.6297057875	0.1954980564
						0.0916714679	0.0916714679	0.4819523024	0.3347047619
						0.2040338880	0.2040338880	0.4996292993	0.0923029247
						0.3483881173	0.3483881173	0.2075502723	0.0956734931
					3	0.2966333890	0.6349633653	0.0684032457	0.0000000000
						0.1439089974	0.8031490682	0.0529419344	0.0000000000
						0.3225890045	0.4968009397	0.1806100558	0.0000000000
10		4				1.0000000000	0.0000000000	0.0000000000	0.0000000000
						0.0678316144	0.0678316144	0.0678316144	0.7965051568
						0.1805746957	0.1805746957	0.1805746957	0.4582759129
						0.3051527124	0.3051527124	0.3051527124	0.0845418628
			3			0.5000000000	0.5000000000	0.0000000000	0.0000000000
						0.3164336236	0.3164336236	0.1835663764	0.1835663764
						0.4219543801	0.4219543801	0.0780456199	0.0780456199
				11		0.9670007152	0.0329992848	0.0000000000	0.0000000000
						0.8922417368	0.1077582632	0.0000000000	0.0000000000
						0.7826176635	0.2173823365	0.0000000000	0.0000000000
						0.6478790678	0.3521209322	0.0000000000	0.0000000000
						0.0265250690	0.0265250690	0.9469498620	0.0000000000
						0.1330857076	0.1330857076	0.7338285848	0.0000000000
						0.4232062312	0.4232062312	0.1535875376	0.0000000000
						0.2833924371	0.2833924371	0.4332151258	0.0000000000
						0.1734555313	0.1734555313	0.5762731177	0.0768158196
						0.0724033935	0.0724033935	0.6893564961	0.1658367169
						0.0768451848	0.0768451848	0.5573732958	0.2889363346
					5	0.3934913008	0.5472380443	0.0592706549	0.0000000000
						0.2462883939	0.6991456238	0.0545659823	0.0000000000
						0.1163195334	0.8427538829	0.0409265838	0.0000000000
						0.2707097521	0.5811217960	0.1481684519	0.0000000000
						0.3019928872	0.4393774966	0.1776946096	0.0809350066

Spectral/hp Element Methods for Computational Fluid Dynamics
George Em Karniadakis and Spencer Sherwin
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APPENDIX E

CHARACTERISTIC FLUX DECOMPOSITION

A hyperbolic conservation law, such as the Euler equations

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0\,, \tag{E.0.1}$$

can be written in non-conservative form:

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0\,, \tag{E.0.2}$$

where $\mathbf{A}(u) = \partial\mathbf{f}/\partial\mathbf{u}$. This form is useful since we can decompose the system with Jacobian matrix $\mathbf{A}$ into its characteristic form to obtain a diagonal matrix of eigenvalues $\mathbf{D}$, that is,

$$\mathbf{L}\cdot\mathbf{A}\cdot\mathbf{R} = \mathbf{D} \quad\Rightarrow\quad \mathbf{A} = \mathbf{R}\cdot\mathbf{D}\cdot\mathbf{L}\,,$$

where $\mathbf{L}$ and $\mathbf{R}$ are the left and right eigenvectors of $\mathbf{A}$ and $\mathbf{RL} = \mathbf{I}$. The non-conservative equation (E.0.2) can therefore be written as

$$\mathbf{u}_t + \mathbf{RDL}\mathbf{u}_x = 0\,.$$

Finally, if we linearise this equation, typically about the Roe average, then we can treat the eigenvector matrices as constant and therefore obtain

$$\mathbf{R}^{-1}\mathbf{u}_t + \mathbf{DL}\mathbf{u}_x = \mathbf{L}\mathbf{u}_t + \mathbf{DL}\mathbf{u}_x = 0\,.$$

This is now a decoupled system in terms of the characteristic variables $\mathbf{v} = \mathbf{L}\mathbf{u}$ to which we are able to apply upwinding techniques or Riemann solvers.

E.1 One dimension

For the one-dimensional Euler system

$$\mathbf{u} = \begin{bmatrix}\rho\\ m\\ E\end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix}\rho u\\ um + p\\ u(p+E)\end{bmatrix},$$

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right),$$

where ρ denotes the density, u is the x component of the velocity, p is the pressure, E is the total energy, $m = \rho u$ is the x component of the momentum, and γ is the ratio of the specific heats of a polytropic gas. The right eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ H-uc & \frac{1}{2}u^2 & H+uc \end{bmatrix},$$

where $c = \sqrt{\gamma p/\rho}$ is the speed of sound, and H represents enthalpy and is defined by

$$H = \frac{E+p}{\rho} = \frac{c^2}{\gamma-1} + \frac{1}{2}u^2.$$

The left eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{L} = \begin{bmatrix} \frac{\frac{1}{2}u^2\beta c + u}{2c} & -\frac{1+u\beta c}{2c} & \frac{\beta}{2} \\ 1-\frac{1}{2}\beta u^2 & \beta u & -\beta \\ \frac{\frac{1}{2}u^2\beta c - u}{2c} & \frac{1-u\beta c}{2c} & \frac{\beta}{2} \end{bmatrix},$$

where $\beta = (\gamma-1)/c^2$. The diagonal matrix of eigenvalues is therefore

$$\mathbf{D} \equiv \mathbf{L}\cdot\mathbf{A}\cdot\mathbf{R} = \begin{bmatrix} u-c & & \\ & u & \\ & & u+c \end{bmatrix}.$$

E.2 Two dimensions

For the two-dimensional Euler equations the same diagonalisation may be applied to the Jacobian matrix $\mathbf{A}(\mathbf{u})$. To obtain a comparable system to eqn (E.0.1) we shall consider the vector $\mathbf{u}$ as being constant in one direction, and so the full Euler system only has x-component derivatives. Therefore we take

$$\mathbf{u} = \begin{bmatrix} \rho \\ m \\ n \\ E \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \rho u \\ um+p \\ un \\ u(p+E) \end{bmatrix},$$

$$p = (\gamma-1)\left(E - \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\right),$$

where ρ, u, E , m, and γ are defined as before, $\mathbf{v} = [u, v]^\top$, and v and $n = \rho v$ are the y components of the velocity and momentum, respectively. The right eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u-c & u & 0 & u+c \\ v & v & 1 & v \\ H-uc & \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) & v & H+uc \end{bmatrix},$$

where $c = \sqrt{\gamma P/\rho}$ is the speed of sound, and H represents enthalpy and is defined by

$$H = \frac{E+p}{\rho} = \frac{c^2}{\gamma-1} + \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}).$$

The left eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{L} = \begin{bmatrix} \frac{\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta c + u}{2c} & -\frac{1+\beta uc}{2c} & -\frac{\beta v}{2} & \frac{\beta}{2} \\ 1-\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta & \beta u & \beta v & -\beta \\ -v & 0 & 1 & 0 \\ \frac{\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta c - u}{2c} & \frac{1-\beta uc}{2c} & -\frac{\beta v}{2} & \frac{\beta}{2} \end{bmatrix},$$

where $\beta = (\gamma-1)/c^2$. The diagonal matrix of eigenvalues is therefore

$$\mathbf{D} \equiv \mathbf{L}\cdot\mathbf{A}\cdot\mathbf{R} = \begin{bmatrix} u-c & & & \\ & u & & \\ & & u & \\ & & & u+c \end{bmatrix}.$$

E.3 Three dimensions

For the three-dimensional Euler equations the same diagonalisation may be applied to the Jacobian matrix $\mathbf{A}(\mathbf{u})$. To obtain a comparable system to eqn (E.0.1) we shall again consider the vector $\mathbf{u}$ as being constant in two directions, and so the full Euler system only has variation in the x direction. Therefore, we consider

$$\mathbf{u} = \begin{bmatrix} \rho \\ m \\ n \\ l \\ E \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \rho u \\ um+p \\ un \\ ul \\ u(p+E) \end{bmatrix},$$

$$p = (\gamma-1)\left(E - \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\right),$$

where ρ, u, v, E, m, n, and γ are defined as before, $\mathbf{v} = [u, v, w]^\top$, and w and $l = \rho w$ are the z components of the velocity and momentum, respectively. The right eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ u-c & u & 0 & 0 & u+c \\ v & v & 1 & 0 & v \\ w & w & 0 & 1 & w \\ H-uc & \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}) & v & w & H+uc \end{bmatrix},$$

where $c = \sqrt{\gamma p/\rho}$ is the speed of sound, and H represents enthalpy and is defined by

$$H = \frac{E+p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{1}{2}(\mathbf{v}\cdot\mathbf{v}).$$

The left eigenvectors of $\mathbf{A}$ can be written as

$$\mathbf{L} = \begin{bmatrix} \frac{\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta c + u}{2c} & -\frac{1+\beta uc}{2c} & -\frac{\beta v}{2} & -\frac{\beta w}{2} & \frac{\beta}{2} \\ 1 - \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta & \beta u & \beta v & \beta w & -\beta \\ -v & 0 & 1 & 0 & 0 \\ -w & 0 & 0 & 1 & 0 \\ \frac{\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})\beta c - u}{2c} & \frac{1-\beta uc}{2c} & -\frac{\beta v}{2} & -\frac{\beta w}{2} & \frac{\beta}{2} \end{bmatrix},$$

where $\beta = (\gamma - 1)/c^2$. The diagonal matrix of eigenvalues is therefore

$$\mathbf{D} \equiv \mathbf{L}\cdot\mathbf{A}\cdot\mathbf{R} = \begin{bmatrix} u-c & & & & \\ & u & & & \\ & & u & & \\ & & & u & \\ & & & & u+c \end{bmatrix}.$$

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