

Review of $\nabla^2 U = 0$ (Boundary Value Problems)

- Laplacian

$$\nabla^2 U(x,y) = \frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} = 0 \quad \left. \right\} \text{(constant coefficients)}$$

$$\nabla^2 U(r,\theta) = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad \left. \right\} \text{(non-constant coefficients)}$$

$$\nabla^2 U(r,\theta,z) = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + U_{zz} = 0$$

$$\nabla^2 U(r,\theta,\phi) = U_{rr} + \frac{2}{r} U_r + \frac{1}{r^2} U_{\theta\theta} + \frac{\cos \theta}{r^2} U_\theta + \frac{1}{r^2 \sin^2 \phi} U_{\phi\phi}$$

- Steady state

$$\int_S \text{Flux} \cdot \hat{n} \, dS = 0 \quad \rightarrow \quad \text{Incoming flux} = \text{Outgoing flux}$$

or

No solution

- Interior Dirichlet $\nabla^2 U = 0$ on a circle

- $U(r,\theta) = r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$

$$q_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta \quad q_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) d\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta$$

- Poisson Integral Formula

$$U(r,\theta) = \text{[shaded region]} \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2} g(\alpha) d\alpha$$

- Annulus Laplacian $\nabla^2 U = 0$

$$U(r,\theta) = a_0 + b_0 \ln r + (a_n r^n + b_n r^{-n}) \cos n\theta + (c_n r^n + d_n r^{-n}) \sin n\theta$$

Coupled coefficients (sets of 2)

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

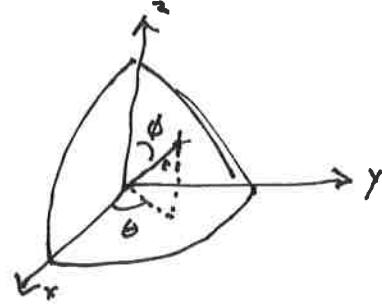
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- Exterior Dirichlet Problem
"Throw out" unbounded r terms

$$U(r, \theta) = r^{-n} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

- Spherical Harmonics (Interior)

$$U(r, \phi) = a_n r^n P_n(\cos \phi)$$



- Spherical Harmonics (Exterior)

$$U(r, \phi) = \frac{a_n}{r^{n+1}} P_n(\cos \phi)$$

- Legendre Polynomials

$$P_0(\cos \phi) = 1, \quad P_1(\cos \phi) = \cos \phi, \quad P_2(\cos \phi) = \frac{1}{2}(3 \cos^2 \phi - 1)$$

Coefficients $a_n = \frac{2m+1}{2} \int_0^\pi g(\phi) P_n(\cos \phi) \sin(\phi) d\phi$

- Green's function

Impulse solution to PDE with BCs. $[G(\text{output}, \text{input})]$

$$U(\text{output}) = \int_{\text{Domain } S} G(\text{output}, \text{input}) f(\text{input}) dS$$

- Superposition and Harmonic Functions (ϕ, ψ)

$$\phi = \frac{1}{2\pi} \theta \quad \psi = V_\infty y \quad \text{identify these?}$$