Brief History of GES 554 (PDEs)

- Motivation

  Laws of Physics, Optimization, Engineering Solutions, etc.

- What is a PDE?

  Derivatives of more than one variable. (e.g. space, time)

- Classification
  - Order: highest derivative in the PDE
  - Variables: independent terms (e.g. x, t)
  - Linear / Non-linear: dependent variable is not mult/div/+/- by du or u.
  - Homogeneous: Zero non-"U" terms.
  - Coefficients: Constant or variable

- Linear PDE classification $\quad Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$
  - Parabolic $(B^2 - 4AC = 0)$ (eg. $u_t = u_{xx}$) heat flow and diffusion
  - Hyperbolic $(B^2 - 4AC > 0)$ (eg. $u_{tt} = u_{xx}$) wave equation
  - Elliptic: $(B^2 - 4AC < 0)$ (eg. $u_{xx} + u_{yy} = 0$) Steady state temp distribution.

Lesson 1

4 types of PDEs seen in this class.

1) Diffusion / Parabolic
   Farlow Lessons 2 - 15

2) Elliptic
   Farlow Lessons 31-36

3) Hyperbolic
   Farlow Lessons 16-30

4) Transport

~~4~~5 Solution Methods seen in this class (not exclusive)

1) Series Expansion

2) Separation of Variables

3) Monte Carlo

4) Finite Difference

5) Ritz /Galerkin

6) Transforms

PDE Tool Box

1) PDE Homogeneous Gov Equ, Homogeneous BCs

Separation of Variables $u = T(t)\,X(x)$

2) PDE, Homogeneous Gov Equ, Non-Homogeneous BCs.

Decompose to $u = \bar{u} + U$ where $\bar{u}$ satisfies BCs

Ex. BC $x(1) = \sin t \Rightarrow \bar{u} = x \sin t$

3) PDE, Non-Homogeneous Gov Equ, Homogeneous BCs.

Decompose to $u = S w$ where S is a solution to the homogeneous problem.

Ex. $u_t = u_{xx} + u$

$u = e^{t} w$

4) PDE, Non-Homogeneous Gov Equ, Homogeneous BCs (Compare with #3)

Eigenfunction Expansion and Series Expansion

$u = \sum T_n(t) X_n(x)$ with X_n from homogeneous problem

Solve for T_n by substitution of u into gov equ.

5) PDE, Infinite Domain

Fourier Transform $-\infty < x < \infty$

Laplace Transform $0 < t < \infty$

ODE Tool Box

- $U' + \lambda U = 0 \rightarrow U = e^{\lambda x}$ 

- $U'' + \lambda^2 U = 0 \rightarrow U = A\sin\lambda x + B\cos\lambda x$

- $U'' - \lambda^2 U = 0 \rightarrow U = A\cosh\lambda x + B\sinh\lambda x$

  or

  $U = Ae^{\lambda x} + Be^{-\lambda x}$

- Non-Homogeneous Gov Equ. Homogeneous BCs.

  Integrating factor. Mult by homogeneous solution term. pull out $\frac{d}{dt}(\cdots + \cdots)$

  Integrate

- Eigenfunctions + Eigenvalues in Sturm-Liouville problems

  Apply generic solution to Boundary Conditions.

  λ_n results from BCs being satisfied.

Laplace. vs Fourier

$$\int_0^\infty f(t)e^{-st}\,dt \qquad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(x)\, e^{-i\xi x}\,dx$$

damping

harmonic

The laplace transform can operate on functions that are not possible with the fourier transform.

- If you see;

  Gov Equ $\quad 0 < x < \infty$, $\quad 0 < t < \infty$

  Think Laplace (mostly)

- If you see

  Gov. Equ $\quad -\infty < x < \infty$, $\quad 0 < t < \infty$

  Think Fourier (mostly)

Fourier Transform

$$F(u_x) = i\xi F(u)$$

$$F(u_{xx}) = -\xi^2 F(u)$$

$$F(u_t) = \frac{d}{dt} F(u)$$

$$F(u_{tt}) = \frac{d^2}{dt^2} F(u)$$

Transform Gov Equ and solve new ODE $F(u)$.

Transform BCs and apply to $F(u)$

Inverse transform of $F(u)$ to give $u(x,t)$

Ex:

$$u_t = \alpha^2 u_{xx}$$

$$F(u_t) = \alpha^2 F(u_{xx}) \Rightarrow \frac{d}{dt} F(u) = \alpha^2 (-\xi^2) F(u)$$

Solve for $F(u) = A e^{-\alpha^2 \xi^2 t}$

Apply BC to find A.

Inverse (lookup in appendix)

$$u = F^{-1}\left(A e^{-\alpha^2 \xi^2 t}\right) = A * F^{-1}\left(e^{-\alpha^2 \xi^2 t}\right)$$

Heat Diffusion

$$U_t = \alpha^2 \nabla^2 U + \text{Source}$$ $\boxed{\text{parabolic}}$

Heat flux is $q = -K\nabla T$ (heat flux flows down the temperature gradient)

Steady State

$U_t = 0$ thus $U_t = \alpha^2 \nabla^2 U + S = 0$

or $\alpha^2 \nabla^2 U + S = 0$ $\boxed{\text{Elliptic}}$

Example:

- find the steady state temperature for $U_t = U_{xx} + x(1-x)$
  $U(0)=0$
  $U(1)=0$

- $U_t = 0 = U_{xx} + x - x^2$

$$\frac{d^2U}{dx^2} = x^2 - x \Rightarrow d^2U = (x^2 - x)\,dx^2$$

- Integrate

$$dU = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)dx + A$$

- Integrate

$$U = \left(\frac{x^4}{12} - \frac{x^3}{6}\right) + Ax + B$$

- Apply BCs

$$0 = 0 - 0 + 0 + B \Rightarrow B = 0$$

$$0 = \left(\frac{1}{12} - \frac{1}{6}\right) + A \Rightarrow A = \frac{1}{12}$$

$$\boxed{U = \frac{x^4}{12} - \frac{x^3}{6} + \frac{1}{12}x}$$

Verify: $U_{xx} = x^2 - x$

$U_{xx} + x - x^2 = 0$ ✓

BCs: $U(0) = 0$ ✓

$U(1) = \frac{1}{12} - \frac{1}{6} + \frac{1}{12} = 0$ ✓

Separation of Variables:

- Convert a PDE of N variables to N ODEs.
- Let $U = X(x)\,Y(y)\,T(t)\,Z(z) \cdots$
- Substitute into the Gov Equ and collect similar variable terms.
- The independent terms must equal a constant.

$$-\lambda^2, \quad n^2, \quad k, \quad n(1-n), \quad \cdots$$

- Solve each ODE and then recombine into U.

Example:

Heat Equation $\quad U_t = \alpha^2 U_{xx} \quad$ with $\quad U(0) = [illegible] \quad U(1) = [illegible]$

$U(x, t=0) = [illegible]$

$$U = X(x)\,T(t)$$

$$X T_t = \alpha^2 X_{xx} T$$

$$\frac{T_t}{\alpha^2 T} = \frac{X_{xx}}{X} = -\lambda^2 \quad \Rightarrow \quad \begin{aligned} T_t + \alpha^2 \lambda^2 T &= 0 \\ X_{xx} + \lambda^2 X &= 0 \end{aligned}$$

$$T = T_0 e^{-\alpha^2 \lambda^2 t}$$

$$X = A \sin \lambda x + B \cos \lambda x$$

Apply BCs to X

$$X(0) = 1 = A \sin 0 + B \cos 0$$

$$X(1) = 1 = A \sin \lambda + B \cos \lambda$$

$$X(x) = A \sin \lambda x \quad \Rightarrow \quad \lambda = n\pi$$

[illegible]

Sine series to fit IC

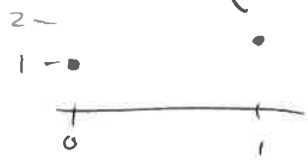
$$\boxed{U = a_n \sin n\pi x \; e^{-\alpha^2 n^2 \pi^2 t}}$$

the high frequency terms die out fast.

Boundary Conditions

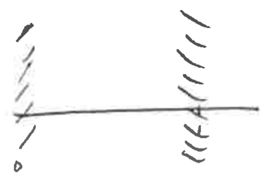
- Dirichlet – Values (think: Dirichlet - Direct)

$u(0) = 1$
$u(1) = 2$



- Neumann – Derivatives

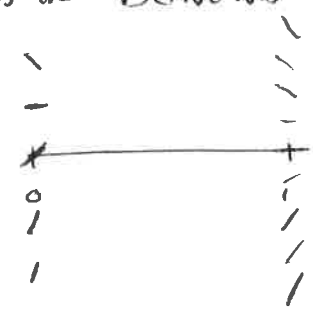
$u_x(0) = 1$
$u_x(1) = -1$



(Needs a ground state)

- Robin – Mixture of Values and Derivatives

$u(0) + u_x(0) = 1$
$u(1) + u_x(1) = 0$



Sturm - Liouville

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

$$a_1 y(0) + a_2 y'(0) = 0$$

$$b_1 y(1) + b_2 y'(1) = 0$$

If your ODE fits that form (and most of ours will), it is a S-L ODE.

- Ordered set of eigenvalues
- Each eigenfunction has an eigenvalue
- All eigenvalues are real
- Eigenfunctions are orthogonal

$$IC = \sum \phi_i$$

Wave Equation

$$U_{tt} = \alpha^2 U_{xx}$$

Needs 2 initial conditions

$$U(x) = f(x)$$
$$U_t(x) = g(x)$$

Reform as 2 1^{st} order equations

$$\frac{d}{dt}\begin{pmatrix} U_x \\ U_t \end{pmatrix} + \begin{bmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{bmatrix} \frac{d}{dx}\begin{pmatrix} U_x \\ U_t \end{pmatrix}$$

eig $A \Rightarrow \lambda = \pm\alpha$ characteristics

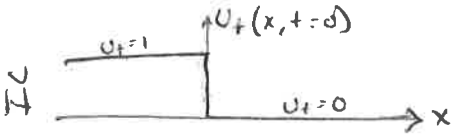
eigvec(A) indicates IC splits into 2 equal pieces

Any displacement IC perturbation is split into 2 parts and moves away at $\pm\alpha$.

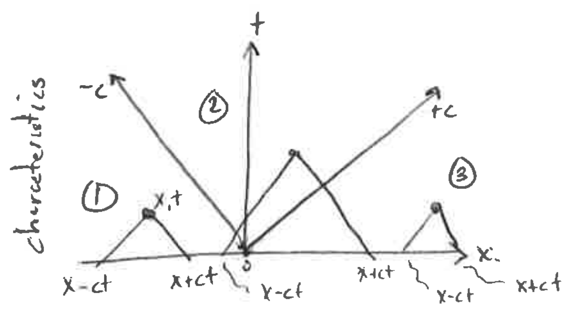
D'Alembert Solution $U_{tt} = c^2 U_{xx}$

$$u(x,t) = \frac{1}{2}\left(f(x-ct)\right) + \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

Example U_t



$c^2 = 4 \Rightarrow c = 2$

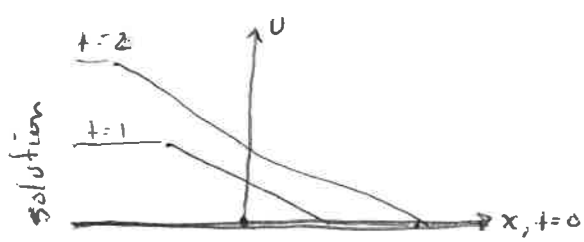


Characteristics divide x−t solution plane into 3 distinct regions. We picked the solution discontinuity as a place to track characteristics.

① $\frac{1}{2 \cdot 2}\int_{x-2t}^{x+2t} (1)\, d\xi = \frac{1}{4}\left(x+2t-x+2t\right) = t$

② $\frac{1}{4}\int_{x-2t}^{0} (1)\, d\xi + \frac{1}{4}\int_{0}^{x+2t} 0\, d\xi = \frac{1}{4}\left(-x+2t\right) = -\frac{x}{4} + \frac{t}{2}$

③ $\frac{1}{4}\int_{x-2t}^{x+2t} 0 = 0$



Strings and Beams

$$U_{tt} = \alpha^2 U_{xx} \quad \text{and} \quad U_{tt} = \alpha^4 U_{xxxx}$$

Separation of Variables works well for both on a finite domain

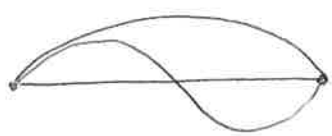
$$T(t) = A \sin \alpha\lambda t + B \cos \alpha\lambda t$$

$$X(x) = A \sin \lambda x + B \cos \lambda x$$

$$T(t) = A \sin \alpha^2\lambda^2 t + B \cos \alpha^2\lambda^2 t$$

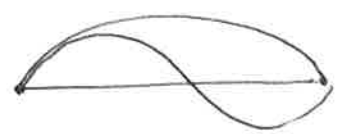
$$X(x) = A \sin \lambda x + B \cos(\lambda x) + C \sinh \lambda x + D \cosh \lambda x$$

Eigenfunctions for Pinned-Pinned



freq $\propto n$

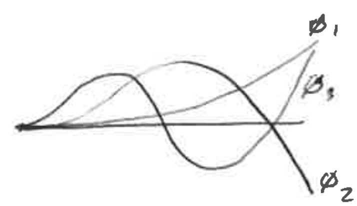
Same shape but different frequencies



freq $\propto n^2$

Eigenfunction for Fixed-free

N/A



Physics Constants

$$U_{tt} = \frac{T}{\mu} U_{xx}$$

$$U_{tt} = \frac{EI}{\mu} U_{xxxx}$$

Boundary Conditions

U

U_x

U	displacement
U_x	slope
U_{xx}	moment
U_{xxx}	shear

Cononical Forms

- Dimensionless Dependent variable

$$U = \overset{*}{U} U_0 \iff \overset{*}{U} = \frac{U}{U_0}$$

often $U_0 = 1\,[\text{unit}]$ is sufficient when BCs are zero. (see Lesson 22)

- Dimensionless Independent variable

$$x = \xi L_0$$
$$t = \tau T_0$$

- Hyperbolic PDEs (Lesson 23) $\quad A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$ with $B^2 - 4AC > 0$

Find characteristics

$$\frac{dy}{dx} = -\frac{d\xi}{dx} \Big/ \frac{d\xi}{dy} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

Find $U_{\xi\eta} = \Phi$ with

$$\bar{A} = 0$$
$$\bar{B} = 2A \xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C \xi_y \eta_y$$
$$\bar{C} = 0$$
$$\bar{D} = \ldots$$
$$\bar{E} = \ldots$$

- Reduce ODEs to canonical form for identification and canned routines

- Parabolic PDEs
  Similar to hyperbolic $\quad U_{\eta\eta} = \Phi$

(Lesson 41)

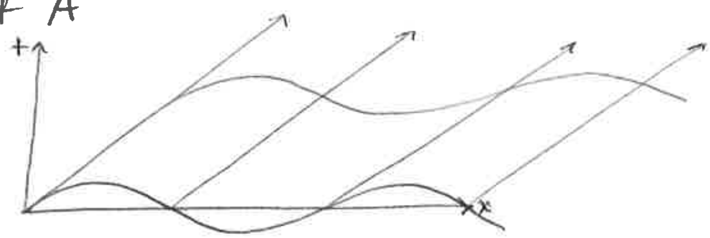
- Elliptic PDEs
  Again similar. $\quad U_{\xi\xi} + U_{\eta\eta} = \Phi$

Characteristics in 1st order PDEs (transport)

$$U_t + A U_x = 0$$

← eig(A) gives characteristics

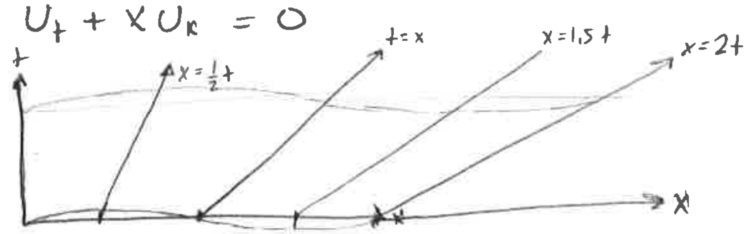
- Scalar PDE,
  - A is the characteristic speed.
  - U is unchanged in value along the characteristic
- Constant A



- Non-Constant.

  A depends on x, t, U.

Ex: $U_t + x U_x = 0$



Constant expansion

IC = $\sin(\pi x)$

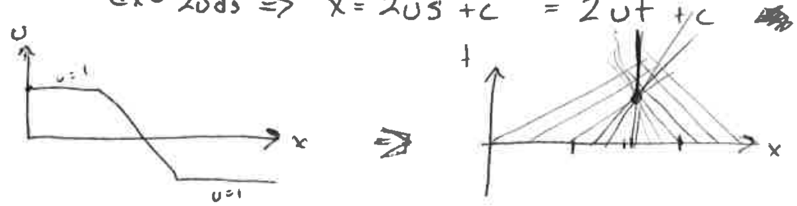
Ex: $U_t + 2UU_x = 0 \Rightarrow U_t + f_x = 0 \qquad f = U^2$

Convert to $\frac{du}{ds} = \frac{du}{dt}\frac{dt}{ds} + \frac{du}{dx}\frac{dx}{ds} = 0$ Compare coefficients $\frac{dt}{ds} = 1$

$\frac{dx}{ds} = 2u$

Thus $dt = ds \Rightarrow t = s + c$

$dx = 2u\,ds \Rightarrow x = 2us + c = 2ut + c$



Characteristic must <u>not</u> cross

Shock speed $s = \dfrac{f(U_R) - f(U_L)}{U_R - U_L}$

Systems of 1st Order PDEs

$$\frac{d}{dt}(\ ) + [A]\frac{d}{dx}(\ ) = 0$$

- eig(A) are characteristics
- eigenvectors give P
- $P^{-1}AP = \Lambda$
- $P\Lambda P^{-1} = A$

Aside:

You should be able to:

- Calculate inverse of 2x2 matrix; flip diagonals, invert off-diagonals, divide by det
- Find eigenvalues of 2x2; $|\lambda I - A| = 0$
- Find eigenvectors of 2x2; $|\lambda I \cdot A| = 0$ with v

Drumheads.

$$c^2 \nabla^2 U = U_{tt}$$

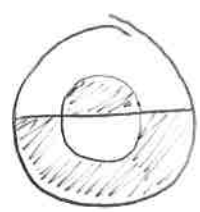
Sep of Vars to space and time terms

$\nabla^2 U + \lambda^2 U = 0$ Helmholtz

$T'' + \lambda^2 c^2 T = 0$ Harmonic motion

Show how Bessel's equation comes from Helmholtz + Sep of Space Vars.

Identify drumhead vibrations and modes.

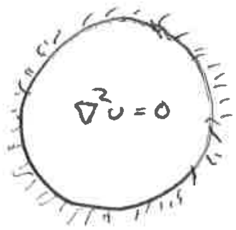


what is this?

Elliptical PDEs.

- Boundary Value problems. Depend <u>only</u> on Gov Equ and BCs.
- Boundary Flux must sum to zero. (not all BCs give solutions)

Interior Dirichlet

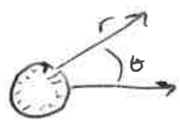


$$U(r,\theta) = r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

Poisson $$U(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \alpha) + r^2}\, g(\alpha)\, d\alpha$$

Value at center is the average of BCs.

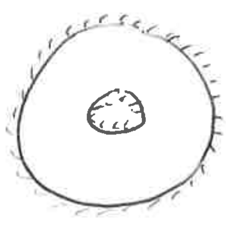
Exterior Dirichlet



$$U(r,\theta) = r^{-n} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

- Could you use a conformal map to ~~[illegible]~~ solve the exterior solution with the interior solution? $w = \frac{1}{z}$

Annulus Dirichlet



$$U(r,\theta) = a_0 + b_0 \ln r + \left(a_n r^n + b_n r^{-n}\right)\cos n\theta + \left(c_n r^n + d_n r^{-n}\right)\sin n\theta$$

Spherical Harmonics

- Constant
- Between spheres
- ϕ only

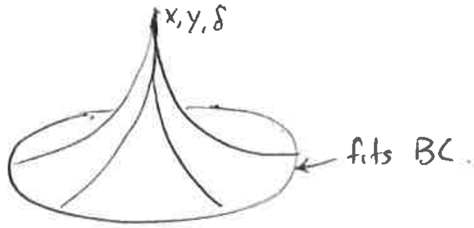
Legendre polynomial

$$U(r,\theta) = a_n r^n P_n(\cos\phi)$$

Green's Function

- A function on a domain that occurs from a pulse input at a location which satisfies the BCs.

- General solution is the summation (integral) of small impulses to match the input function.

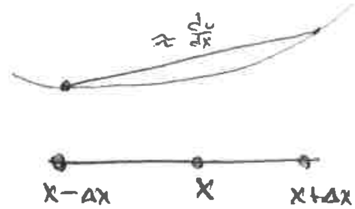


- $U(\cancel{x}, \cancel{y}) = \int\int G(x, y, \alpha, \beta)\, f(\alpha, \beta)\, d\alpha\, d\beta$

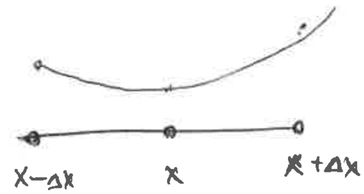
- Powerful when you can <u>find</u> a Green's function.

Finite Difference Approx.

$$\frac{du}{dx} \approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$



$$\frac{d^2u}{dx^2} \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$



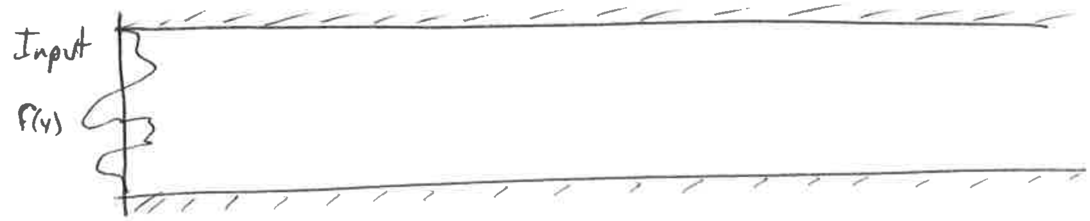
Explicit Methods

Use current derivatives to estimate future value

Implicit Methods

Use future derivatives to search for future value.

2D Waveguide



$$p'(x,y,t) = \underbrace{\frac{2}{d}\int_0^d f(y)\cos\left(\frac{n\pi y}{d}\right)dy}_{\text{BC Input}} \; \underbrace{\cos\frac{n\pi y}{d}}_{\text{y-dir}} \; \underbrace{e^{\pm i\left(\frac{\omega^2}{a_0^2} - \frac{n^2\pi^2}{d^2}\right)^{1/2} x}}_{\text{x-dir}} \; \underbrace{e^{-i\omega t}}_{\text{Harmonic in time}}$$

When $\frac{\omega^2}{a_0^2} > \frac{n^2\pi^2}{d^2}$ propogation in x-dir

$\frac{\omega^2}{a_0^2} < \frac{n^2\pi^2}{d^2}$ decay in x-dir

$\frac{\omega^2}{a_0^2} = \frac{n^2\pi^2}{d^2}$ cuton / cutoff frequency

phase shift of velocity when cuton

Calc of Variations

Minimize $J(y) = \int_a^b F(x,y,y')\,dx$

$$\frac{dF}{dy} - \frac{d}{dx}\left(\frac{dF}{dy'}\right) = 0$$

Ritz Method.

Minimize functional to solve PDE:

Ex:

Solve $U_{xx} = f$ on $0 < x < 1$ $\quad u(0) = 0$, $u(1) = 1$, $f = 1$

$$J = \int_0^1 \left(U_x^2 + 2uf\right)dx$$

$$U = a_1 \phi_1 + a_2 \phi_2 = a_1 x + a_2 x^2$$

$$U_x = a_1 + 2a_2 x$$

$$J = \int_0^1 (a_1 + 2a_2 x)^2 + 2(a_1 + 2a_2 x)\; dx$$

$$= \int_0^1 \left(a_1^2 + 4a_1 a_2 x^2 + 4a_2^2 x^2 + 2a_1 + 4a_2 x\right) dx$$

$$= \cancel{\int} \left(a_1^2 x + \frac{4}{3} a_1 a_2 x^3 + \frac{4}{3} a_2^2 x^3 + 2a_1 x + \frac{4}{2} a_2 x^2\right)\Big|_0^1 dx$$

$$= a_1^2 + \frac{4}{3} a_1 a_2 + \frac{4}{3} a_2^2 + 2a_1 + 2a_2$$

$$\frac{dJ}{da_1} = 2a_1 + \frac{4}{3} a_2 + \cancel{} + 2 = 0$$

$$\frac{dJ}{da_2} = \frac{4}{3} a_1 + \frac{8}{3} a_2 + 2 = 0$$

Solve for a_1 and a_2

$$\begin{bmatrix} 2 & 4/3 \\ 4/3 & 8/3 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

Ritz.

ϕ = set of functions

Minimize energy functional

Example:

$$F = U_x^2 + U_y^2 + 2Uf \quad \text{in} \quad J(U) = \int_\Omega F \, d\Omega$$

for

$$U_{xx} + U_{yy} = f$$

A theoretical basis for finite element formulations.

Perturbation Methods

- slightly change the PDE with a free variable
- Match terms and solve resulting linear PDE

Conformal Mapping

Solve on an easier domain and map 1:1 to a more difficult domain

Powerful with $\nabla^2 U = 0$ and 2D domains

Follow ~~up~~ on Classes

Harmonic functions, Green's functions → Complex Vars

Heat Diffusion → Heat Transfer ME 309

Aerodynamics ϕ, ψ → Aero I, II, grad AEM500
Airfoil + Wing Theory
Flow Control

Numerical Solutions → Num Analysis AEM or CS or EE

Num Solutions to PDEs of large systems → Finite Elements

Visualization of Solutions → CS or my lab

Waves + Vibration → Adv Struct Dynamics
~~CFD~~
Acoustics, Unsteady Aero

Transport Equations → CFD, Aero, Visc Flows
ChemE classes