Multistage Rocket Performance
Project Two

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ABSTRACT

Formulas for obtaining the thrust of a rocket engine are given. A computer program is used to find the combustion products. A computer program is adapted and used to solve for a rocket's velocity and distance for two stages when given basic rocket parameters. Data is tabulated and plotted.

INTRODUCTION

Rocket performance depends on geometry, fuel types and flight location. Assumptions of adiabatic combustion and frozen ideal gas products and the use of simple rocket equations yield an easy method of calculating the performance of a rocket.

The objective is to determine the flight properties of a two stage rocket given fuel types and basic nozzle geometry. A computer program will be used to calculate the product species and temperatures. Isentropic flow assumptions will be used to find the properties of the fluid. Thrust will then be calculated and differentially solved to yield the velocity and altitude of a rocket.

THEORY

Rockets provide thrust by accelerating a fluid. From Newton's law, $m \cdot \frac{dV}{dt} = \Sigma F$. From a force balance,

$$m\frac{dV}{dt} = Thrust - Drag - gravity$$

The thrust can be consided as the sum of a pressure and a flow contribution. ($Thrust = \dot{m} \cdot V_e + (P_{exit} - P_{atmo}) \cdot A_e$) For a rocket carring its own fuel, the current mass is the initial mass, m_o, minus the burned fuel, $\dot{m} \cdot t$. ($m = m_o - \dot{m} \cdot t$) Solving the force balance for dV/dt and neglecting the drag term yields,

$$\frac{dV}{dt} = \frac{\dot{m} \cdot V_e}{m_o - \dot{m}t} + \frac{A_e}{m_o - \dot{m} \cdot t} \cdot (P_{exit} - P_{atm}) - g$$

This is an second order ordinary differential equation.

The flow of a fluid in a rocket exhaust can be modeled by assuming isentripic flow, ideal gas and constant specific heat. A overall product of combustion has properties of the mean of the individual products. Thus, the mean mixture molecular weight is

$$M = \Sigma(x_i M_i) = \Sigma(\frac{n_i}{n_t} M_i)$$

where n_i is the number of moles, n_t is the total product moles and M_i is the individual product's molecular weight. Similarly, the overall specific heat is

$$C_p(T) = \Sigma(C_{p_i}\frac{n_i}{n_t})$$

From the molecular weight and specific heat, the specific gas constant and ratio of specific heats can be calculated.

$$R = \frac{\bar{R}}{M}$$

$$\gamma = \frac{C_p}{C_p - R}$$

Due to the complexities and small time, the product ratios will be assumed to remain constant. Assuming frozen species amounts allows for ease of fluid properties throughout the nozzle flow.

A relationship between parameters of an oblique shock wave is derived by Curle and can be solved for the oblique shock angle,

$$\frac{P_2 - P_1}{P_1} = \frac{2\gamma}{\gamma+1}M_1^2 sin^2\theta - \frac{2\gamma}{\gamma+1}$$

Thus,

$$\theta = arcsin\sqrt{\Big(\frac{P_2 - P_1}{P_1} + \frac{2\gamma}{\gamma+1}\Big)\Big(\frac{\gamma+1}{2\gamma M_1^2}\Big)}$$

METHOD OF CALCULATION

Product species and temperatures were found from a FORTRAN 77 program, AFTC2, when given combustion pressure, reactants and the heat of combustion. Dissociation of the products was assumed. The adiabatic temperature was used in all calculations. Fitted curves for the properties of the products were used to calculate the Molecular weight and specific heat at the chamber temperature. Gamma was calculated. Exit properties were calculated using tables given in Appendix D. Thrust at sea level, design height and space were calculated from the above theory. A Second order Runge-Kutta numerical solver given in Appendix E was used to find the velocity and altitude versus time when given the net forces acting on the rocket. More than one stage is implemented by running the solver routine again with different parameters but with the final velocity and altitude from the end of the previous stage. The program outputs incremental velocities at each time step. After the final stage, the program exits.

RESULTS AND DISCUSSION

Calculations were performed as discussed above and are given in Appendix A,B and C. Stage one has a total mass of 5000 kg, a structure weight of 500 kg and a propellent mass of 3500 kg for a mass loading of 0.7. Stage two has a total mass of 1000 kg (5000-3500-500=1000), a propellent mass of 500 kg and a 500 kg payload for a mass loading of 0.5. An exit area of 0.05 m^2 was chosen for stage two.

The first stage burned Oxygen and Hydrogen. At the stoichiometric mixture and assuming dissociation, the adiabatic flame temperature was 4003 K (Appendix C). This yields a gas constant of 0.5155 $\frac{KJ}{kgK}$ and a gamma of 1.196. The back pressure was approximated as 30.09 Kpa at the design height of 9.1 kilometers. The combustion chamber had a pressure of 4.05 MPa. The products exiting the nozzle have a Mach number of 3.55, a temperature of 1791.6 K and a velocity of 3731 m/s. The mass flow rate was 25.95 kg/s with an exit plane area of 2135 cm^2. At the design altitude, the thrust was 96.8 KN. At ground level, the thrust was 81.7 KN with an oblique shock angle of 30.07 degrees. In space, the thrust was 103.2 KN with a Prandl-Meyer turn angle of 132.6 degrees. A plot of thrust versus altitude for the first stage is given in Appendix G.

The second stage burned Kerosene and Oxygen. A stoichiometric mixture with dissociation yielded an adiabatic flame temperature of 3955 K. The exit Mach number is 3.58 which is 2941 m/s. The second stage has a mass flow rate of 7.75 kg/s.

At approximately 135 seconds into the flight, the first stage separates and the second stage starts. Plots of altitude and velocity versus time are given in the Figures in Appendix G. The first stage has a burn length of 3500kg / 25.95 kg/s = 134.9 seconds and achieves a velocity of 3282 m/s at 150000 m. (Appendix G) The second stage has a burn length of 500 kg / 7.75 kg/s = 64.5 s. This stage adds another 1538 m/s of velocity and 250000 m of altitude. This shows how lighter upper stages are able to accelerate the rocket to similar velocities and distances will significatly less fuel. This is due to the smaller thrust being applied to a much smaller mass. To accelerate the upper stage, the lower stage must be significatly larger and any increase in the mass of the fuel cuts into the payload capability.

CONCLUSION

Assumptions of adiabatic combustion and frozen ideal gas products and the use of simple rocket equations were able to roughly calculate the performance of a rocket. A multistage rocket is sensitive to the proportion of weight divided between the stages. The analysis could easily add friction and changing gravitational effects due to the ODE being numerically solved. This method to calculate the performance of a rocket is only approximate; however, it is suitable for a rough solution or for experiments to obtain insights into rocket performance.

REFERENCES

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Modern Fluid Dynamics, Van Nostrand Reinhold Company, London.

APPENDIX A

Calculations: Hydrogen and Oxygen

Combustion products:

From theory, the mean mixture molecular weight is $MW = \frac{\Sigma n_i MW_i}{n_t}$. The molar species amounts are given in the computer output in Appendix C. Thus,

$$MW = \frac{(0.7672)(18) + (0.1164)(32) + (0.2328)(2)}{(1.116)} = 16.129\frac{kg}{kmol}$$

The specific heats were evaluated from a fitted line curve (Lilley) at the chamber temperature. The constant pressure specific heat is from theory,

$$C_p = \frac{\Sigma C_{p_i} n_i}{n_t \cdot MW} = \frac{(55.63)(.7672) + (41.55)(.1164) + (39.32)(.2328)}{(1.116)(16.129)} = 3.1483\frac{KJ}{KgK}$$

The specific gas constant, R, is equal to $\frac{\bar{R}}{MW} = \frac{8.314}{16.129} = 0.5155\frac{KJ}{KgK}$. The ratio of specific heats is

$$\gamma = \frac{C_p}{C_p - R} = \frac{3.1483}{3.1483 - 0.5155} = 1.196$$

Nozzle:

A logistic line-fit between zero and 30 kilometers was used to find an approximation for the atmospheric pressure given the altitude in kilometers. The routine returned

$$P_{atmo} = \frac{326.5}{1 + 2.23e^{0.1636h}}$$

where h is in kilometers and P is in KPa. At the design height, 9.1 km, the pressure is $\frac{326.5}{1+2.23e^{0.1636 \cdot 9.1}} = 30.09Kpa$. The Chamber Pressure is 4050 Kpa. Since the chamber pressure is also the stagnation pressure, $P/P_t = \frac{30.09}{4050} = 0.00743$. Thus, the exit Mach number can be found by interpolating the Gamma = 1.196 isentropic flow table given in Appendix D for the calculated P/P_t. The Mach number at the exit is 3.55. T/T_t at M=3.55 is 0.4475 so that the exit temperature is (4003.5)(0.4475) = $1791.6K$. A/A_* is 15.143. The exit area is $(A/A_*)(A_*) = (15.143)(141) = 2135cm^2$. The exit velocity is

$$V_e = \sqrt{\gamma RT} \cdot M = \sqrt{(1.196)(0.5155)(1791.6)(1000)} \cdot 3.55 = 3731\frac{m}{s}$$

The mass flow rate calculated with exit values is

$$\dot{m} = \rho VA = \frac{P}{RT}VA = \frac{(30.09)}{(0.5155)(1791.6)}(3731)(.2135) = 25.95\frac{kg}{s}$$

Performance:

The general thrust for a rocket is $Th = \dot{m} \cdot V_e + (P_e - P_a)A$. In space with no outside pressure, Thrust is $(25.95)(3731) + (30.09 - 0)(1000)(.21352) = 103.2KN$. At the design altitude, 9.1 km, the back pressure is equal to the atmospheric pressure so that $Th = (25.95)(3731) + 0 = 96.8KN$. At sea level, the atmospheric pressure is 101 Kpa. The thrust is $(25.95)(3731) + (30.09 - 101)(1000)(.21352) = 81.7KN$.

The Prandl-Meyer angle is given by (John)

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}} arctan \sqrt{\frac{\gamma+1}{\gamma-1}(M^2-1)} - arctan\sqrt{M^2-1}$$

Space pressure is zero so that the fluid attempts an infinite increase in velocity. Thus, $\nu(\infty) = 211.25°$. At the exit Mach number, $\nu(3.55) = 78.66°$. Thus, the Prandl-Meyer Turn Angle is 211.25-78.66=132.6 degrees.

At ground level, an oblique shock is present behind the exit plane. From theory,

$$\theta = arcsin \sqrt{\Big(\frac{101 - 30.09}{30.09} + \frac{2 \cdot 1.196}{1.196+1}\Big)\Big(\frac{1.196+1}{2 \cdot 1.196(3.55)^2}\Big)} = 30.07°$$

APPENDIX B

Calculations: Kerosene and Oxygen

Molecular Weight

From theory, the mean mixture molecular weight is $MW = \frac{\Sigma n_i MW_i}{n_t}$. The molar species amounts are given in the computer output in Appendix C. Thus,

$$MW = \frac{(6.418)(28) + (3.582)(44) + (9.028)(18) + (4.195)(32) + (1.972)(2)}{(25.19)} = 25.33 \frac{kg}{kmol}$$

The specific heats were evaluated from a fitted line curve (Lilley) at the chamber temperature, 3955 K. The constant pressure specific heat is from theory,

$$C_p = \frac{\Sigma C_{p_i} n_i}{n_t \cdot MW}$$

$$= \frac{(6.418)(37.72) + (3.592)(64.29) + (9.028)(55.57) + (4.195)(41.497) + (1.972)(39.24)}{(25.33)(25.19)}$$

$$= 1.921 \frac{KJ}{KgK}$$

The specific gas constant, R, is equal to $\frac{\bar{R}}{MW} = \frac{8.314}{25.33} = 0.3282 \frac{KJ}{KgK}$. The ratio of specific heats is

$$\gamma = \frac{C_p}{C_p - R} = \frac{1.921}{1.921 - 0.3282} = 1.206$$

Nozzle:

From the previous stage, A/A_* is 15.143. Thus, the exit Mach number can be found by interpolating the Gamma = 1.206 isentropic flow table given in Appendix D for the known A/A_*. The Mach number at the exit is 3.58. T/T_t at M=3.58 is 0.4311 so that the exit temperature is (3956)(0.4311) = $1705K$. The exit pressure is found by multiplying the interpolating P/P_t at M=3.58 and the chamber pressure. $P_e = (4050)(0.00728) = 29.5 Kpa$ The exit velocity is

$$V_e = \sqrt{\gamma RT} \cdot M = \sqrt{(1.206)(0.3282)(1705)(1000)} \cdot 3.58 = 2941 \frac{m}{s}$$

The exit area is 0.05 m^2. The mass flow rate calculated with exit values is

$$\dot{m} = \rho V A = \frac{P}{RT} V A = \frac{(29.5)}{(0.3282)(1705)}(2941)(0.05) = 7.75 \frac{kg}{s}$$

APPENDIX C

Computer Output:

Product Species and Temperature

APPENDIX D

Isentropic Flow Tables:

APPENDIX E

Computer Program:

2nd Order ODE Numerical Solver

APPENDIX F

Computer Program:

Numerical Solver Output

APPENDIX G

Figures